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GENERATING NOVEL PROBABILITY DISTRIBUTIONS: A UD FRACTIONAL DERIVATIVE APPROACH

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SUMMARY

This study introduces novel probability distributions derived from the Dixit and Ujlayan (UD) fractional differential equation. By applying the UD fractional differential equation to established continuous probability distributions, new probability distributions are formulated. The resulting UD fractional probability distributions extend classical distributions such as the gamma, power function, arcsine, and beta distributions, thereby expanding the theoretical framework for probability modeling. An application of the UD fractional Beta distribution to a real-world dataset demonstrates its superior flexibility and adaptability compared to the classical Beta distribution, particularly in modeling skewed and bounded data. These findings underscore the potential of UD fractional distributions in addressing complex data modeling challenges across diverse fields.

Keywords and phrases: Conformable fractional derivative, fractional differential equation, probability distribution, UD derivative

AMS Classification: 26A33; 60E05.

1 Introduction and Motivation

In data modeling, identifying optimal probability distributions is a key objective. Previous studies, such as those by Stansell (2004) and Tayfun (1980), have demonstrated the effectiveness of the Rayleigh distribution in modeling ocean wave heights. Similarly, Martins et al. (2017) employed the Pareto distribution to model extreme data, including the highest one-day rainfall and ecological population sizes.

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Researchers continuously seek new methods to generate probability distributions that offer greater flexibility in data fitting. Notable developments include the skew normal distribution introduced by Azzalini (1985), which adds a skewness parameter to the normal distribution, and the exponentiated Weibull distribution proposed by Mudholkar and Srivastava (1993), enhancing the Weibull distribution's flexibility. For further insights, see Marshall and Olkin (1997), Gleaton and Lynch (2006), Mahdavi and Kundu (2017), and related works.

Differential equations have long served as foundational tools for generating probability distributions. Pearson (1895) made a pioneering contribution by introducing a family of distributions derived from a specific differential equation, leading to what is now known as the Pearson system of distributions. The Pearson distribution, denoted as $f(x)$, is defined as any solution to the following differential equation:

$$\frac{f'(x)}{f(x)} + \frac{x-d}{ax^2+bx+c} = 0,$$

where a, b, c, and d represent distributional parameters. The solution to this differential equation is expressed as:

$$f(x) = A\exp\left\{\int \frac{d-x}{ax^2+bx+c}dx\right\},$$

with A being the constant of integration. The emergence of the normal distribution occurs when $a = b = 0$, with a mean of $-d$ and a variance of c. Further exploration of distributions producible through this differential equation can be found in works such as Pearson (1895), Pearson (1901), Pearson (1916), and Johnson et al. (1994).

The development of alternative probability distributions can be attributed to the work of Burr (1942). The Burr probability distribution, denoted as $f(x) = F'(x)$, is defined as any solution to the following differential equation:

$$\frac{F'(x)}{F(x)\,(1-F(x))} - g(x) = 0,$$

where $g(x)$ is a non-negative function ensuring that $F(x)$ increases over its support and $0 \leq F(x) \leq 1$. The solution to this differential equation is given by:

$$F(x) = \frac{1}{1+e^{-\int g(x)dx}},$$

with the specific form of the solution depending on the choice of $g(x)$. Further insights into Burr-type distributions resulting from this differential equation can be found in works such as Johnson et al. (1994), Burr (1942), and Fry (1993).

Fractional differential equations extend the concept of ordinary differential equations and have been widely discussed in the literature. Various definitions for fractional derivatives have been proposed, with the Riemann-Liouville and Caputo formulations being the most recognized.

Riemann-Liouville Definition: For $\alpha \in [n-1, n)$, the α-derivative of a function f is expressed as:

$$D^{\alpha}_{t_0}(f)(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}\frac{f(x)}{(t-x)^{\alpha-n+1}}dx.$$

Caputo Definition: Similarly, for $\alpha \in [n-1, n)$, the α-derivative of f is defined as:

$$D_{t_0}^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

These definitions serve as fundamental tools in the study of fractional calculus, offering distinct perspectives on fractional derivatives and facilitating their application in various scientific domains.

However, these definitions have certain limitations:

1. The Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1) = 0$, whereas the Caputo definition successfully addresses this issue.

2. Both definitions do not adhere to the product rule of differentiation: $D_a^{\alpha}(fg) \neq f\left(D_a^{\alpha} g\right) + g\left(D_a^{\alpha} f\right)$.

3. Similarly, the quotient rule of differentiation is not satisfied by these definitions: $D_a^{\alpha}\left(\frac{f}{g}\right) \neq \frac{g(D_a^{\alpha} f) - f(D_a^{\alpha} g)}{g^2}$.

4. The chain rule is not preserved by these definitions: $D_a^{\alpha}(f \circ g) \neq f^{(\alpha)}(g(t)) g^{(\alpha)}(t)$.

5. The index rule is not universally valid: $D^{\alpha} D^{\beta}(f) \neq D^{\alpha+\beta}(f)$, presenting a general discrepancy.

These challenges highlight the complexity of fractional derivatives and the importance of choosing an appropriate definition based on the specific mathematical or scientific context.

The introduction of a novel fractional derivative, termed the conformable fractional derivative, is credited to the work of Khalil et al. (2014). For $0 < \alpha < 1$, the conformable fractional derivative of a function f, denoted as T_{α}, is formally defined as:

$$T_{\alpha} f(t) = \lim_{h \to 0} \frac{f\left(t + h t^{1-\alpha}\right) - f(t)}{h}.$$

This definition represents a natural extension of the first-order derivative. Alternatively, it can be expressed as:

$$T_{\alpha} f(t) = t^{1-\alpha} f'(t),$$

highlighting its relationship with the first derivative. Importantly, the conformable fractional derivative is designed to satisfy nearly all the classical properties associated with the usual first derivative. This conceptualization offers a promising framework for fractional calculus, demonstrating its compatibility with established principles of differentiation.

Recently, conformable fractional differential equations have been applied in the creation of novel probability distributions. Notably, Hammad et al. (2020) introduced various conformable fractional probability distributions, encompassing the fractional exponential distribution, fractional Lomax distribution, fractional Levy distribution, fractional Rayleigh distribution, and fractional gamma distribution. The specific characteristics of the conformable fractional Rayleigh probability distribution and the conformable gamma distribution were thoroughly examined by Jebril et al. (2021a) and Jebril et al. (2021b), respectively. Furthermore, a more recent study conducted by Amleh et al. (2022) delved into the conformable Lomax probability distribution, exploring its properties and distinctive features. These investigations underscore the emerging role of conformable fractional differential equations in advancing the understanding and application of probability distributions.

Dixit and Ujlayan, as documented in Dixit et al. (2020) and Dixit and Ujlayan (2021), introduced the UD fractional derivative, represented by $\left(D^{\alpha} f\right)(t) = (1-\alpha) f(t) + \alpha f'(t)$, where $\alpha \in (0, 1]$. This formulation reflects a convex combination of the function and its first derivative. Moreover, their work extensively explores the fundamental properties and outcomes associated with this fractional differential operator.

The complexity of computing derivatives of arbitrary order, particularly in mathematical modeling involving fractional derivatives, has led us to adopt the UD fractional derivative. Its computational simplicity, analytical tractability, and alignment with classical derivative properties make it an attractive choice. Consequently, the UD fractional differential operator mitigates the intricacies associated with calculating solutions for fractional ordered differential equations, presenting an analytical approach. The proposed definition, by transforming a fractional derivative into a convex combination of the function and its ordinary derivative without introducing additional variables, further simplifies the equation. Ultimately, this reduction allows for the application of well-established methods for solving classical differential equations.

Alhribat and Samuh (2023) presented a new approach to generating fractional continuous probability distributions by solving UD fractional differential equations associated with well-known continuous probability distributions such as Exponential, Pareto, Lomax, and Levy distributions. Their research concludes that UD fractional differential equations offer a promising technique for generating continuous fractional probability distributions, expanding the parameter space of existing distributions and potentially leading to better fits for real-world data.

This article employs the UD fractional differential equation, as defined by Dixit et al. (2020) and Dixit and Ujlayan (2021), to generate new fractional distributions based on established probability distributions.

The subsequent sections are organized as follows: Section 2 introduces the UD fractional derivative and elucidates its fundamental properties. Sections 3.1 to 3.4 derive UD fractional probability distributions for the gamma, power function, Arcsine, and beta distributions, respectively. Section 4 visualizes the UD fractional Gamma distribution, while Section 5 applies the UD fractional Beta distribution to real-world data. Finally, Section 6 concludes the study with remarks on key findings and implications.

2 UD Fractional Derivative and Fundamental Properties

In this section, we introduce the definition of the UD fractional derivative and elucidate some of its key properties, drawing primarily from the work of Dixit and Ujlayan (2021).

Definition 2.1. For a given function $f : [0, \infty) \to \mathbb{R}$ and $\alpha \in [0, 1]$, the UD derivative of order α is defined as:

$$D^{\alpha} f(x) = \lim_{\epsilon \to 0} \frac{e^{\epsilon(1-\alpha)} f\left(x e^{\frac{\epsilon\alpha}{x}}\right) - f(x)}{\epsilon}. \tag{2.1}$$

If this limit exists, then $D^{\alpha} f(x)$ is termed the UD derivative of f for $\alpha \in [0,1]$, with the understanding that $D^{\alpha} f(x) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}$. Additionally, if f is UD differentiable in the interval $(0,x)$ for $x > 0$ and $\alpha \in [0,1]$ such that $\lim_{x\to 0^+} f^{\alpha}(x)$ exists, then:

$$f^{\alpha}(0) = \lim_{x \to 0^+} f^{\alpha}(x).$$

Theorem 1. *Let $f : [0,\infty) \to \mathbb{R}$ be a differentiable function, and $\alpha \in [0,1]$. Then, f is UD differentiable, and the UD derivative is expressed as:*

$$D^{\alpha} f(x) = (1-\alpha) f(x) + \alpha f'(x). \tag{2.2}$$

Proof. By Definition 2.1, we have:

$$\begin{aligned}
D^{\alpha} f(x) &= \lim_{\varepsilon \to 0} \frac{e^{\varepsilon(1-\alpha)} f\left(x e^{\frac{\varepsilon\alpha}{x}}\right) - f(x)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\left\{1 + \varepsilon(1-\alpha) + o\left(\varepsilon^2\right)\right\}\left[f\left\{x + \varepsilon\alpha + o\left(\varepsilon^2\right)\right\}\right] - f(x)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\{1 + \varepsilon(1-\alpha)\}\left[f(x) + f'(x)\{\varepsilon\alpha\}\right] - f(x)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{f(x) + \varepsilon(1-\alpha) f(x) + \varepsilon\alpha f'(x) - f(x)}{\varepsilon} \\
&= (1-\alpha) f(x) + \alpha f'(x),
\end{aligned}$$

where $\alpha \in [0,1]$.

Remark 1. The UD derivatives of order $\alpha, \alpha \in [0,1]$, for some elementary real-valued differentiable functions in $[0,\infty)$, can be expressed as follows:

1. $D^{\alpha}(\lambda) = (1-\alpha)\lambda$ for all constants $\lambda \in \mathbb{R}$.
2. $D^{\alpha}\left((ax+b)^n\right) = (1-\alpha)(ax+b)^n + an\alpha(ax+b)^{n-1}$ for all $a, b \in \mathbb{R}$.
3. $D^{\alpha}\left(e^{ax+b}\right) = ((1-\alpha) + a\alpha)e^{ax+b}$ for all $a, b \in \mathbb{R}$.
4. $D^{\alpha}(\sin(ax+b)) = (1-\alpha)\sin(ax+b) + a\alpha\cos(ax+b)$ for all $a, b \in \mathbb{R}$.
5. $D^{\alpha}(\cos(ax+b)) = (1-\alpha)\cos(ax+b) - a\alpha\sin(ax+b)$ for all $a, b \in \mathbb{R}$.
6. $D^{\alpha}(\log(ax+b)) = (1-\alpha)\log(ax+b) + a\alpha(ax+b)^{-1}$ for all $a, b \in \mathbb{R}$.

Theorem 2. *Let f and g be two differentiable functions in $[0,\infty)$ and $0 \leq \alpha, \gamma \leq 1$, then the following properties hold:*

1. *Linearity: $D^{\alpha}(af + bg) = aD^{\alpha} f + bD^{\alpha} g$ for all $a, b \in \mathbb{R}$.*

2. *Product rule:* $D^{\alpha}(fg) = (D^{\alpha}f)\,g + \alpha(Dg)f$.

3. *Quotient rule:* $D^{\alpha}\left(\frac{f}{g}\right) = \frac{(D^{\alpha}f)g - \alpha(Dg)f}{g^2}$*, provided* $g(x) \neq 0$ *for all* $x \in [0, \infty)$.

4. *Chain rule:* $D^{\alpha}(f \circ g)(t) = (1-\alpha)(f \circ g)(t) + \alpha f'(g(t))g'(t)$. *The classical chain rule* $D_a^{\alpha}(f \circ g) = f^{(\alpha)}(g(t))g^{(\alpha)}(t)$ *does not apply here.*

5. *Commutativity:* $D^{\alpha}(D^{\gamma})\,f = D^{\gamma}(D^{\alpha})\,f$.

Proof. The proof of the first four parts is straightforward. Here is the proof of Part 5. Using Equation 2.2, we get:

$$\begin{aligned} D^{\alpha}(D^{\gamma})\,f &= (1-\alpha)(1-\gamma)f + \alpha(1-\gamma)f' + \gamma(1-\alpha)f' + \alpha\gamma f'' \\ &= D^{\gamma}(D^{\alpha})\,f. \end{aligned}$$

This completes the proof.

Remark 2. The UD derivative of order $\alpha, \alpha \in [0,1]$, as given in Definition 2.1, violates Leibnitz's rule for fractional derivatives, $D^{\alpha}(fg) \neq gD^{\alpha}f + fD^{\alpha}g$. It also violates the law of indices, $D^{\alpha}(D^{\gamma})\,f \neq D^{\alpha+\gamma}f$.

Definition 2.2. A fractional derivative D^{α} has a conformable property if $D^{\alpha}(t) \to f'(t)$ when $\alpha \to 1$.

Remark 3. Equation 2.2 establishes that the UD derivative of order α, where $\alpha \in [0,1]$, applied to a differentiable function $f : [0,\infty) \to \mathbb{R}$, assumes a convex combination structure involving the function and its first derivative. Specifically, $D^{\alpha}f(x) = f(x)$ for $\alpha = 0$ and $D^{\alpha}f(x) = f'(x)$ for $\alpha = 1$, demonstrating the conformable property inherent in the UD derivative.

Theorem 3. *Let f be a bounded function defined on* $[0,\infty)$*. If f is UD differentiable for some* $\alpha \in [0,1]$ *at $x = a$, then f is continuous at $x = a$.*

Proof. To establish continuity at $x = a$, it suffices to demonstrate that $\lim_{\epsilon\to 0} f(x+\epsilon\alpha) = f(x)$.

$$\begin{aligned} &\lim_{\epsilon\to 0} f(x+\epsilon\alpha) - f(x) \\ &= \lim_{\epsilon\to 0}\left(\frac{(1+\epsilon(1-\alpha))f(x+\epsilon\alpha) - \epsilon(1-\alpha)f(x+\epsilon\alpha) - f(x)}{\epsilon}\right)\epsilon \\ &= \lim_{\epsilon\to 0}\left(\frac{(1+\epsilon(1-\alpha))f(x+\epsilon\alpha) - f(x)}{\epsilon}\right)\epsilon - \lim_{\epsilon\to 0}\epsilon(1-\alpha)f(x+\epsilon\alpha) \\ &= \lim_{\epsilon\to 0}(D^{\alpha}f)\,\epsilon - \lim_{\epsilon\to 0}\epsilon(1-\alpha)f(x+\epsilon\alpha) \\ &= 0\,(\text{since } f \text{ is bounded for all } 0 \leq x < \infty) \end{aligned}$$

Consequently, fractional UD Calculus finds application in probability theory, offering a valuable tool for determining the structural characteristics of probability distributions and facilitating parameter estimation. Specifically, the continuous probability density function can be formulated as an ordinary differential equation, the solution of which yields the probability density function. Conversely, a UD fractional version of this ordinary differential equation can be solved to obtain a new fractional continuous probability distribution, aligning with the original distribution when $\alpha = 1$.

3 Some UD Fractional Distributions

3.1 UD fractional Gamma distribution

The Gamma distribution is defined by the probability density function:

$$f(x) = \frac{x^{k-1}e^{-\frac{x}{\theta}}}{\Gamma(k)\theta^k}, \quad x > 0,\ k, \theta > 0.$$

Let $y = \frac{x^{k-1}e^{-\frac{x}{\theta}}}{\Gamma(k)\theta^k}$. The first derivative of y is given by:

$$y' = \left(\frac{-1}{\theta} + \frac{k-1}{x}\right)\frac{x^{k-1}e^{-\frac{x}{\theta}}}{\Gamma(k)\theta^k} = \left(\frac{-1}{\theta} + \frac{k-1}{x}\right)y.$$

This leads to the first-order ordinary differential equation:

$$y' - \left(\frac{-1}{\theta} + \frac{k-1}{x}\right)y = 0.$$

Now, let's consider the α-order differential equation involving the UD derivative:

$$y^{(\alpha)} - \left(\frac{-1}{\theta} + \frac{k-1}{x}\right)y = 0$$
$$(1-\alpha)y + \alpha y' - \left(\frac{-1}{\theta} + \frac{k-1}{x}\right)y = 0,$$

which simplifies to:

$$\alpha y' = \left(\alpha - 1 + \frac{-1}{\theta} + \frac{k-1}{x}\right)y$$
$$\frac{y'}{y} = \frac{\alpha-1}{\alpha} - \frac{1}{\alpha\theta} + \frac{k-1}{\alpha x}.$$

As a result, we obtain:

$$\ln y = -\left(\frac{1+\theta-\alpha\theta}{\alpha\theta}\right)x + \frac{k-1}{\alpha}\ln x + c$$
$$y = Ax^{\frac{k-1}{\alpha}}e^{-\left(\frac{1+\theta-\alpha\theta}{\alpha\theta}\right)x}, \qquad \text{where } A = e^c > 0$$

Hence, the updated probability distribution is given by:

$$f_\alpha(x) = Ax^{\frac{k-1}{\alpha}}e^{-\left(\frac{1+\theta-\alpha\theta}{\alpha\theta}\right)x},$$

where the normalization constant A can be determined by solving the integral equation:

$$\int_0^\infty f_\alpha(x)dx = 1.$$

This yields the expression:

$$\frac{1}{A} = \Gamma\left(\frac{k-1+\alpha}{\alpha}\right)\left(\frac{1+\theta-\alpha\theta}{\alpha\theta}\right)^{-\frac{k-1+\alpha}{\alpha}}, \quad k > 1-\alpha,\ 1+\theta > \alpha\theta.$$

Ultimately, the modified probability distribution can be expressed as:

$$f_\alpha(x) = \frac{x^{\frac{k-1}{\alpha}} e^{-\left(\frac{1+\theta-\alpha\theta}{\alpha\theta}\right)x}}{\Gamma\left(\frac{k-1+\alpha}{\alpha}\right)\left(\frac{1+\theta-\alpha\theta}{\alpha\theta}\right)^{-\frac{k-1+\alpha}{\alpha}}}, \quad k > 1-\alpha,\ 1+\theta > \alpha\theta,\ 0 < \alpha \leq 1,\ \theta > 0.$$

It is evident that $f_\alpha(x)$ reverts to the Gamma distribution. It is noteworthy that:

$$\lim_{\alpha \to 1^-} f_\alpha(x) = \frac{x^{k-1}e^{-\frac{x}{\theta}}}{\Gamma(k)\theta^k} = f(x).$$

3.2 UD fractional power function distribution

A random variable X is considered to follow a Power function distribution if its probability density function is expressed as:

$$f(x;\lambda,\beta) = \frac{\lambda}{\beta}\left(\frac{x}{\beta}\right)^{\lambda-1}, \quad 0 < x < \beta,\ \lambda, \beta > 0.$$

Let $y = \frac{\lambda}{\beta}\left(\frac{x}{\beta}\right)^{\lambda-1}$. The first derivative of y is calculated as:

$$y' = \frac{(\lambda-1)}{\beta}\left(\frac{x}{\beta}\right)^{-1}\left(\frac{\lambda}{\beta}\left(\frac{x}{\beta}\right)^{\lambda-1}\right) = \frac{(\lambda-1)}{\beta}\left(\frac{x}{\beta}\right)^{-1} y.$$

This leads to the following first-order ordinary differential equation:

$$y' - \frac{(\lambda-1)}{\beta}\left(\frac{x}{\beta}\right)^{-1} y = 0.$$

Now, let's consider the α-order differential equation with respect to the UD derivative:

$$y^{(\alpha)} - \frac{(\lambda-1)}{\beta}\left(\frac{x}{\beta}\right)^{-1} y = 0$$

$$(1-\alpha)y + \alpha y' - \frac{(\lambda-1)}{\beta}\left(\frac{x}{\beta}\right)^{-1} y = 0,$$

which simplifies to:

$$\alpha y' = \left(\alpha - 1 + \frac{(\lambda-1)}{\beta}\left(\frac{x}{\beta}\right)^{-1}\right) y$$
$$\frac{y'}{y} = \frac{\alpha-1}{\alpha} + \frac{(\lambda-1)}{\beta\alpha}\left(\frac{x}{\beta}\right)^{-1}.$$

This results in the expression:

$$\ln y = \left(\frac{\alpha-1}{\alpha}\right) x + \left(\frac{\lambda-1}{\alpha}\right) \ln\left(\frac{x}{\beta}\right) + c$$
$$y = A\left(\frac{x}{\beta}\right)^{\frac{\lambda-1}{\alpha}} e^{\frac{\alpha-1}{\alpha}x}, \qquad \text{where } A = e^c > 0.$$

Thus, the new probability distribution will be:

$$f_\alpha(x) = A\left(\frac{x}{\beta}\right)^{\frac{\lambda-1}{\alpha}} e^{\frac{\alpha-1}{\alpha}x},$$

where the normalizing constant A can be found by solving the integral equation:

$$\int_0^\beta f_\alpha(x)dx = 1.$$

Using Equation (3.383) in Gradshteyn and Ryzhik (2007), the constant A is expressed as:

$$A = \frac{1}{\beta \mathbf{B}\left(1, \frac{\lambda+\alpha-1}{\alpha}\right) {}_1F_1\left(\frac{\lambda+\alpha-1}{\alpha}; \frac{\lambda+\alpha-1}{\alpha}+1; \left(\frac{\alpha-1}{\alpha}\right)\beta\right)}$$
$$= \frac{\lambda+\alpha-1}{\beta\alpha \; {}_1F_1\left(\frac{\lambda+\alpha-1}{\alpha}; \frac{\lambda+\alpha-1}{\alpha}+1; \left(\frac{\alpha-1}{\alpha}\right)\beta\right)}, \quad \alpha \leq \lambda + 1,$$

where $\mathbf{B}(\cdot,\cdot)$ and ${}_1F_1(\cdot;\cdot;\cdot)$ are the Beta and confluent hypergeometric functions, respectively (Gradshteyn and Ryzhik, 2007). The Beta function is defined as:

$$\mathbf{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad x>0,\, y>0,$$

or equivalently:

$$\mathbf{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(\cdot)$ is the Gamma function. The confluent hypergeometric function ${}_1F_1(a;b;x)$, also known as Kummer's function, is given by:

$${}_1F_1(a;b;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} x^n,$$

where $(a)_n$ and $(b)_n$ are the Pochhammer symbols (rising factorials).

It is worth noting that the restriction $(\lambda > 1-\alpha)$ is necessary for the integration to be convergent. However, for the fractional derivative, α must be between 0 and 1, which is a subset of $(\lambda > 1-\alpha)$. Thus, the integration is also convergent over $(0 < \alpha < 1)$. Finally, the new probability distribution can be expressed as:

$$f_\alpha(x) = \frac{(\lambda+\alpha-1)\left(\frac{x}{\beta}\right)^{\frac{\lambda-1}{\alpha}} e^{\frac{\alpha-1}{\alpha}x}}{\beta\alpha\ {}_1F_1\left(\frac{\lambda+\alpha-1}{\alpha}; \frac{\lambda+\alpha-1}{\alpha}+1; \left(\frac{\alpha-1}{\alpha}\right)\beta\right)}, \quad 0 < x < \beta,\ \alpha > 1-\lambda,\ \beta, \lambda > 0.$$

It is evident that $f_\alpha(x)$ is again a Power function distribution. Note that:

$$\lim_{\alpha\to 1^-} f_\alpha(x) = \frac{\lambda}{\beta}\left(\frac{x}{\beta}\right)^{\lambda-1} = f(x).$$

3.3 UD fractional Arcsine distribution

The Arcsine distribution is characterized by the probability density function:

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \quad 0 < x < 1.$$

Let $y = \frac{1}{\pi\sqrt{x(1-x)}}$. The first derivative of y is determined as:

$$y' = \frac{x-(1-x)}{2\pi(x(1-x))^{\frac{3}{2}}} = \frac{x-(1-x)}{2x(1-x)}y.$$

This results in the following first-order ordinary differential equation:

$$y' - \frac{x-(1-x)}{x(1-x)}y = 0.$$

Now, consider the α-order differential equation with respect to the UD derivative:

$$y^{(\alpha)} - \frac{x-(1-x)}{2x(1-x)}y = 0$$
$$(1-\alpha)y + \alpha y' - \frac{x-(1-x)}{2x(1-x)}y = 0,$$

then:

$$\alpha y' = \left(\alpha - 1 + \frac{1}{2(1-x)} - \frac{1}{2x}\right)y$$
$$\frac{y'}{y} = \frac{\alpha-1}{\alpha} + \frac{1}{2\alpha(1-x)} - \frac{1}{2\alpha x}.$$

This leads to the expression:

$$\ln y = \left(\frac{\alpha-1}{\alpha}\right)x - \frac{1}{2\alpha}\ln(1-x) - \frac{1}{2\alpha}\ln x + c$$
$$y = Ax^{\frac{-1}{2\alpha}}(1-x)^{\frac{-1}{2\alpha}}e^{\frac{\alpha-1}{\alpha}x}, \qquad \text{where } A = e^c > 0$$

Thus, the new probability distribution is given by:

$$f_\alpha(x) = Ax^{\frac{-1}{2\alpha}}(1-x)^{\frac{-1}{2\alpha}}e^{\frac{\alpha-1}{\alpha}x},$$

where the normalizing constant A is determined by solving the integral equation:

$$\int_0^1 f_\alpha(x)dx = 1.$$

Using Equation (3.383) in Gradshteyn and Ryzhik (2007),

$$A = \frac{1}{\mathbf{B}\left(\frac{2\alpha-1}{2\alpha}, \frac{2\alpha-1}{2\alpha}\right)\ {}_1F_1\left(\frac{2\alpha-1}{2\alpha}; \frac{2\alpha-1}{\alpha}; \frac{\alpha-1}{\alpha}\right)}, \quad 0 < \alpha < 1.$$

It is noteworthy to observe that the restriction ($\alpha \leq \lambda + 1$) is necessary for the integration to be convergent. However, for the fractional derivative, α must be between 0 and 1, which is a subset of ($\alpha \leq \lambda + 1$). Thus, the integration is also convergent over ($0 < \alpha < 1$). Finally, the new probability distribution can be expressed as:

$$f_\alpha(x) = \frac{x^{\frac{-1}{2\alpha}}(1-x)^{\frac{-1}{2\alpha}}e^{\frac{\alpha-1}{\alpha}x}}{\mathbf{B}\left(\frac{2\alpha-1}{2\alpha}, \frac{2\alpha-1}{2\alpha}\right)\ {}_1F_1\left(\frac{2\alpha-1}{2\alpha}; \frac{2\alpha-1}{\alpha}; \frac{\alpha-1}{\alpha}\right)}, \quad 0 < x < 1,\ 0 < \alpha < 1.$$

It is evident that $f_\alpha(x)$ is again an Arcsine Distribution. Note that:

$$\lim_{\alpha \to 1^-} f_\alpha(x) = \frac{1}{\pi\sqrt{x(1-x)}} = f(x).$$

3.4 UD fractional Beta distribution

The Beta distribution is defined as:

$$f(x; \lambda, \beta) = \frac{x^{\lambda-1}(1-x)^{\beta-1}}{\mathbf{B}(\lambda, \beta)}, \quad 0 < x < 1,\ \lambda > 0,\ \beta > 0.$$

Let $y = \frac{x^{\lambda-1}(1-x)^{\beta-1}}{\mathbf{B}(\lambda,\beta)}$, and the first derivative of y is calculated as:

$$\begin{aligned} y' &= \left(-(\beta-1)(1-x)^{-1} + (\lambda-1)x^{-1}\right)\frac{x^{\lambda-1}(1-x)^{\beta-1}}{\mathbf{B}(\lambda,\beta)} \\ &= \left(\frac{-(\beta-1)}{(1-x)} + \frac{(\lambda-1)}{x}\right)y \end{aligned}$$

This leads to the first-order ordinary differential equation:

$$y' - \left(\frac{-(\beta-1)}{(1-x)} + \frac{(\lambda-1)}{x}\right)y = 0.$$

Now, consider the α-order differential equation with respect to the UD derivative:

$$y^{(\alpha)} - \left(\frac{-(\beta-1)}{(1-x)} + \frac{(\lambda-1)}{x} \right) y = 0$$
$$(1-\alpha)y + \alpha y' - \left(\frac{-(\beta-1)}{(1-x)} + \frac{(\lambda-1)}{x} \right) y = 0,$$

then:

$$\alpha(1-x)xy' - ((\alpha-1)(1-x)x + (\lambda-1)(1-x) + (1-\beta)x)y = 0.$$

Hence:

$$\frac{y'}{y} = \frac{(\alpha-1)}{\alpha} + \frac{(1-\beta)}{\alpha(1-x)} + \frac{(\lambda-1)}{\alpha x}$$

This yields:

$$\ln y = \frac{(\alpha-1)x}{\alpha} + \frac{(\beta-1)\ln(1-x)}{\alpha} + \frac{(\lambda-1)\ln x}{\alpha} + c$$
$$\ln y = \frac{(\alpha-1)x}{\alpha} + \ln(1-x)^{\frac{\beta-1}{\alpha}} + \ln x^{\frac{\lambda-1}{\alpha}} + c$$
$$y = Ae^{\frac{(\alpha-1)x}{\alpha}}(1-x)^{\frac{\beta-1}{\alpha}} x^{\frac{\lambda-1}{\alpha}}, \quad \text{where } A = e^c > 0.$$

Setting $f_\alpha(x) = Ae^{\frac{(\alpha-1)x}{\alpha}}(1-x)^{\frac{\beta-1}{\alpha}} x^{\frac{\lambda-1}{\alpha}}$. For $f_\alpha(x)$ to be a PDF, we require $\int_0^1 f_\alpha(x)dx = 1$. Thus:

$$\int_0^1 Ae^{\frac{(\alpha-1)x}{\alpha}}(1-x)^{\frac{\beta-1}{\alpha}} x^{\frac{\lambda-1}{\alpha}} dx = 1.$$

This leads to:

$$A\int_0^1 e^{\frac{(\alpha-1)x}{\alpha}}(1-x)^{\frac{\beta-1}{\alpha}} x^{\frac{\lambda-1}{\alpha}} dx = 1.$$

Using Equation (3.383) in Gradshteyn and Ryzhik (2007),

$$A\mathbf{B}\left(\frac{\lambda+\alpha-1}{\alpha}, \frac{\beta+\alpha-1}{\alpha} \right) {}_1F_1\left(\frac{\lambda-1+\alpha}{\alpha}; \frac{\beta+\lambda-2+2\alpha}{\alpha}; \frac{\alpha-1}{\alpha} \right) = 1.$$

This results in:

$$A = \frac{1}{\mathbf{B}\left(\frac{\lambda+\alpha-1}{\alpha}, \frac{\beta+\alpha-1}{\alpha} \right) {}_1F_1\left(\frac{\lambda-1+\alpha}{\alpha}; \frac{\beta+\lambda-2+2\alpha}{\alpha}; \frac{\alpha-1}{\alpha} \right)}.$$

Finally, the new probability distribution is expressed as:

$$f_\alpha(x) = \frac{e^{\frac{(\alpha-1)x}{\alpha}}(1-x)^{\frac{\beta-1}{\alpha}} x^{\frac{\lambda-1}{\alpha}}}{\mathbf{B}\left(\frac{\lambda+\alpha-1}{\alpha}, \frac{\beta+\alpha-1}{\alpha} \right) {}_1F_1\left(\frac{\lambda-1+\alpha}{\alpha}; \frac{\beta+\lambda-2+2\alpha}{\alpha}; \frac{\alpha-1}{\alpha} \right)},$$
$$x \in (0,1),\ \alpha \in (0,1],\ \lambda, \beta > 1-\alpha.$$

It is evident that $f_\alpha(x)$ is again a Beta distribution. Note that:

$$\lim_{\alpha \to 1^-} f_\alpha(x) = \frac{x^{\lambda-1}(1-x)^{\beta-1}}{\mathbf{B}(\lambda, \beta)} = f(x).$$

4 Visualization of the UD Fractional Gamma Distribution

To provide a comprehensive understanding of the UD fractional Gamma distribution, this section focuses on visualizing its PDF and CDF for various parameter configurations. While this analysis specifically highlights the UD fractional Gamma distribution, similar methodologies can be applied to visualize other UD fractional distributions, such as the UD fractional Beta, Power Function, and Arcsine distributions. The UD fractional Gamma distribution is governed by three parameters: α (fractional parameter), θ (scale parameter), and k (shape parameter), which collectively determine its shape and scale. By varying these parameters, we demonstrate the flexibility of the distribution and its ability to model diverse probabilistic behaviors. Figure 1 illustrates the PDF and CDF curves for different combinations of α, θ, and k, highlighting the impact of parameter changes on the distribution's behavior.

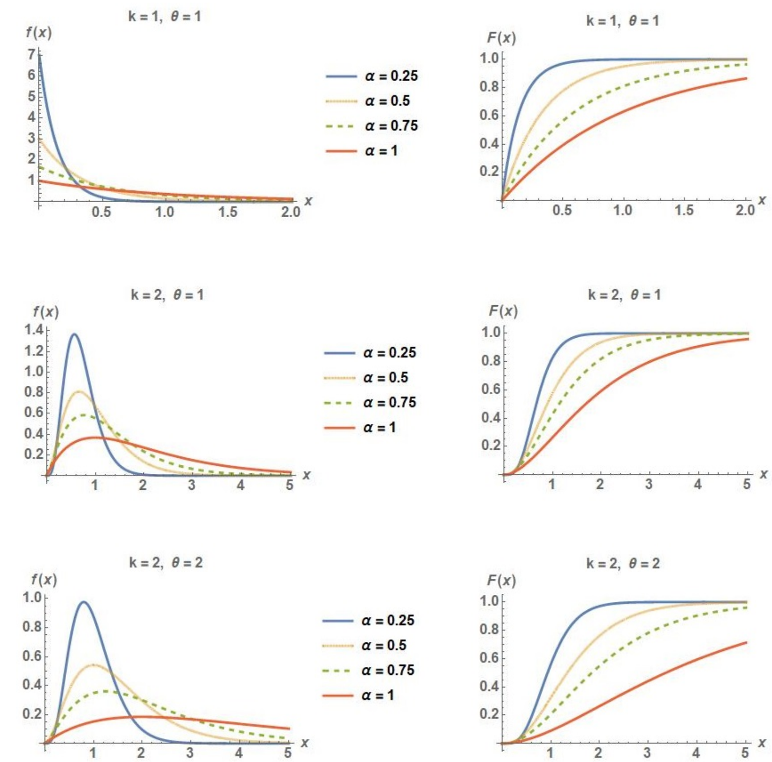

Figure 1: PDF (left) and CDF (right) curves of the UD fractional Gamma distribution for varying α, θ, and k. The plots highlight the flexibility of the distribution as parameters change.

The following concluding remarks are drawn from the visualizations:

- **Effect of the fractional parameter, α:**

  - As α increases from $\alpha = 0.25$ to $\alpha = 1$, the PDF becomes less peaked and spreads out more. For $\alpha = 1$, the distribution reverts to the classical Gamma distribution.
  - The CDF curves indicate that smaller values of α lead to faster accumulation of probabilities, reflecting a more concentrated distribution around smaller values.

- **Effect of the shape parameter, k:**
  - When $k = 1$, the PDF exhibits behavior similar to an exponential distribution, characterized by monotonic decay.
  - For $k = 2$, the PDF becomes unimodal, resembling the classical Gamma distribution, with a prominent peak. The CDF shows steeper increases with higher k, indicating reduced skewness and faster convergence to 1.

- **Effect of the scale parameter, θ:**
  - Increasing θ (e.g., from $\theta = 1$ to $\theta = 2$) stretches the PDF horizontally, resulting in a lower peak height while maintaining the overall shape.
  - Similarly, the CDF shifts rightward, indicating more gradual probability accumulation as θ increases.

The analysis highlights the flexibility of the UD fractional Gamma distribution in adapting to various data patterns through its parameters. This adaptability enhances its potential for modeling complex phenomena in diverse fields.

5 Application to Real Data

In this section, we demonstrate the practical relevance of the proposed UD fractional distributions by applying the UD fractional Beta distribution to a real dataset. While the methodology can be extended to other distributions such as UD fractional Gamma, Power Function, and Arcsine distributions, this analysis primarily focuses on the Beta distribution for illustrative purposes. The study assesses the goodness-of-fit of the UD fractional Beta distribution and compares it with the classical Beta distribution, emphasizing the flexibility and advantages of the UD fractional Beta in modeling real-world data.

We conduct an analysis on a dataset relevant to SAR (Synthetic Aperture Radar) image modeling, specifically examining the visibility of oil slicks in oceanic environments. This dataset, obtained from Iqbal et al. (2021), comprises 120 observations representing intensity values captured during SAR imaging. These values, ranging between 0 and 1, display a skewed-right distribution, making them conducive to modeling using fractional Beta distributions.

Figure 2 presents a boxplot summarizing the distribution of the dataset. It highlights key characteristics such as skewness and central tendency, providing an effective visual representation of the data.

This dataset exemplifies a continuous, skewed distribution commonly encountered in SAR image analysis. The skewness and bounded nature of the data underscore the limitations of classical models

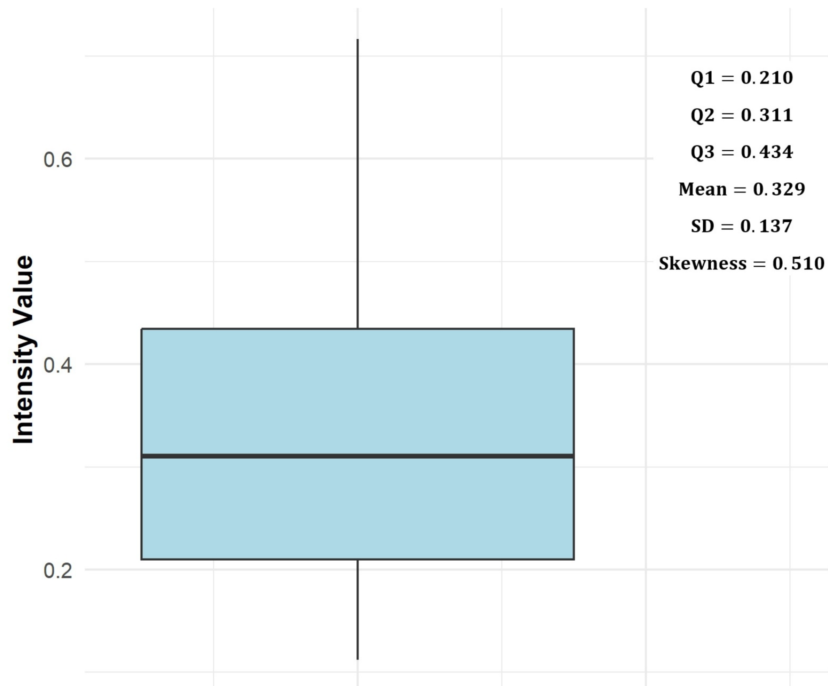

Figure 2: Boxplot of the SAR dataset showing the skewness and key descriptive statistics. The figure includes quartiles (Q1 = 0.210, Q2 = 0.311, Q3 = 0.434), mean (0.329), standard deviation (0.137), and skewness (0.510).

such as Gaussian distributions, emphasizing the necessity for more adaptable distributions like the proposed UD fractional Beta.

Parameter Estimation:

- Maximum likelihood estimation (MLE), known for its asymptotic efficiency and robustness, was employed to estimate the parameters α, β, and λ of the UD fractional Beta distribution.

- Similarly, for the classical Beta distribution, MLE was used to estimate the shape parameters β and λ.

Log-Likelihood Function: The log-likelihood function for the UD fractional Beta distribution is expressed as:

$$\begin{aligned}\ell(\alpha, \lambda, \beta; x_1, \ldots, x_n) &= \sum_{i=1}^{n} \log f_\alpha(x_1, \ldots, x_n; \alpha, \lambda, \beta) \\ &= \sum_{i=1}^{n} \left[\frac{(\alpha-1)x_i}{\alpha} + \frac{\beta-1}{\alpha} \log(1-x_i) + \frac{\lambda-1}{\alpha} \log(x_i) \right. \\ &\quad \left. - \log \mathbf{B}\left(\frac{\lambda+\alpha-1}{\alpha}, \frac{\beta+\alpha-1}{\alpha} \right) - \log {}_1F_1\left(\frac{\lambda-1+\alpha}{\alpha}; \frac{\beta+\lambda-2+2\alpha}{\alpha}; \frac{\alpha-1}{\alpha} \right) \right],\end{aligned}$$

where $f_\alpha(x_1, x_2, \ldots, x_n; \alpha, \lambda, \beta)$ is the PDF of the UD fractional Beta distribution.

Model Fit Assessment:

- **Visual Fit**: The empirical density function of the data was compared against the fitted PDF of both the UD fractional Beta and classical Beta distributions. The comparison reveals that the UD fractional Beta distribution provides a better fit.
- **Goodness-of-Fit Statistics**: Various metrics were computed to evaluate and compare the two models, including the Kolmogorov-Smirnov test statistic, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC).

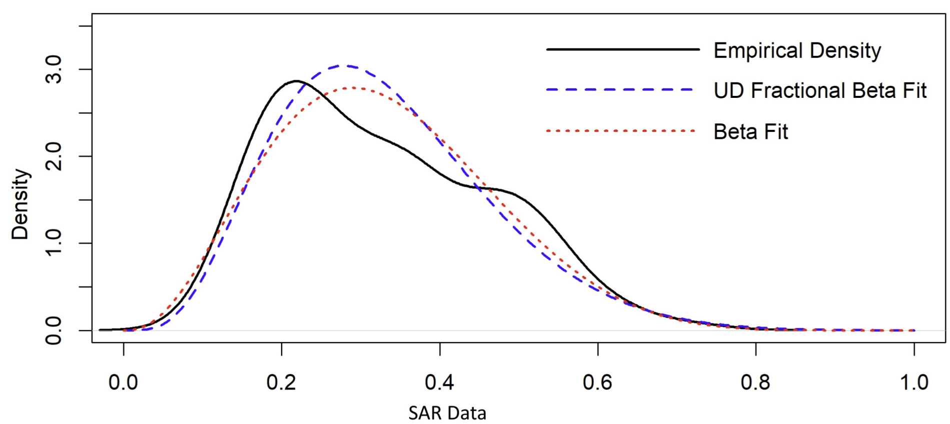

Figure 3: Empirical density function of the data overlaid with the fitted PDFs of the UD fractional Beta and classical Beta distributions.

The analysis of the real dataset using the UD fractional Beta and classical Beta distributions offers valuable insights into the suitability and performance of these models. Table 1 summarizes the parameter estimates for both distributions. For the UD fractional Beta distribution, the estimated parameters are $\alpha = 0.0798$ (SE = 0.0537), $\beta = 1.1761$ (SE = 0.3723), and $\lambda = 1.3237$ (SE =

Table 1: Estimated parameters for the UD fractional Beta and classical Beta distributions.

Distribution	**Parameter**	**Estimate**	**Standard Error**
UD fractional Beta	α	0.0798	0.0537
	β	1.1761	0.3723
	λ	1.3237	0.1423
Classical Beta	λ	3.8177	0.4361
	β	7.7484	0.9222

Table 2: Goodness-of-Fit metrics for the UD fractional Beta and classical Beta distributions.

Metric	**UD fractional Beta**	**Classical Beta**
Kolmogorov-Smirnov p-value	0.4922	0.4904
Akaike Information Criterion (AIC)	-145.3151	-145.3140
Bayesian Information Criterion (BIC)	-136.9527	-139.7390

0.1423). In contrast, the classical Beta distribution yields estimates of $\lambda = 3.8177$ (SE = 0.4361) and $\beta = 7.7484$ (SE = 0.9222).

Furthermore, Table 2 showcases the goodness-of-fit metrics for the UD fractional Beta and classical Beta distributions. The Kolmogorov-Smirnov p-value for the UD fractional Beta is 0.4922, slightly higher than the value of 0.4904 for the classical Beta, indicating a marginal improvement in the fit for the UD fractional Beta model. Additionally, the AIC and BIC values provide further insight into model efficiency, balancing goodness of fit with model complexity. The AIC of -145.3151 for the UD fractional Beta is lower than the value of -145.3140 for the classical Beta, suggesting that the UD fractional Beta distribution offers a slightly better fit when considering the trade-off between fit and parsimony. However, the BIC values of -136.9527 and -139.7390 for the UD fractional Beta and classical Beta, respectively, suggest that the classical Beta distribution may be preferred when applying a stricter penalty for model complexity. These results highlight the nuanced performance of these models and the importance of considering multiple criteria in model selection.

In conclusion, based on the estimated parameters and goodness-of-fit metrics, the UD fractional Beta distribution demonstrates superior performance in modeling the skewed and bounded real-world data compared to the classical Beta distribution. The flexibility and adaptability of the UD fractional Beta model make it a valuable tool for analyzing datasets with similar characteristics. These findings suggest the potential advantages of using the UD fractional Beta distribution in applications where skewed and complex data distributions are prevalent.

6 Conclusion

In this study, we have leveraged the UD fractional differential equation as a potent instrument for constructing novel continuous fractional probability distributions. The fundamental concept involves deriving the fractional differential equation from an established (baseline) probability distribution characterized by k parameters. The solution to this differential equation yields a (new) probability distribution featuring $k + 1$ parameters. Notably, the resulting distribution may either align with the same distribution family as the baseline, as exemplified by the Gamma distribution, or diverge into an entirely distinct family.

The introduction of UD fractional distributions, such as the fractional Beta distribution, highlights their flexibility and adaptability in modeling complex real-world data. The application to a real dataset demonstrated the superior performance of the UD fractional Beta distribution over the classical Beta distribution in capturing skewed and bounded data, underscoring the practical relevance of these distributions. Such improvements can significantly enhance data modeling capabilities in various domains, including environmental analysis, finance, and biomedical research.

The theoretical contributions of this study lay a strong foundation for further advancements in the field. Future research could explore the extension of UD fractional differential equations to multivariate distributions or dynamic systems, providing tools for modeling interactions or time-dependent behaviors. Additionally, the computational efficiency of these distributions could be further optimized, enabling their application to larger datasets or real-time analysis.

Another promising direction involves integrating UD fractional distributions with machine learning algorithms, where these flexible distributions can serve as priors or model components in Bayesian frameworks. Such integration may improve the interpretability and robustness of predictive models.

In conclusion, the UD fractional derivative offers a versatile framework for developing new probability distributions that address the limitations of classical models. The findings of this study highlight the potential of these methods in tackling complex data modeling challenges, paving the way for innovative applications and theoretical developments in statistics and applied mathematics.

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