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ON WEIGHTED GENERALIZED AKASH DISTRIBUTION WITH PROPERTIES AND ITS APPLICATIONS OF REAL LIFETIME DATA USING R PROGRAMMING

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SUMMARY

In this paper, we introduce a new class of generalized Akash distribution with suitable R code known as weighted generalized Akash distribution has been established. The developed new distribution has been generated by using the weighted technique to its baseline distribution and has been described with its specific statistical features and characteristics. Furthermore, its parameters have been estimated based upon the technique of maximum likelihood estimation. Finally, a real life data set has been investigated and fitted to demonstrate the applicability and flexibility of a new distribution.

Keywords and phrases: Weighted distribution, Generalized Akash distribution, Survival analysis, Order statistics, Maximum likelihood estimation.

1 Introduction

The theory of probability distributions has played a crucial role in probability and statistics for modeling, analyzing, and interpreting complicated different lifetime data sets that occur from diversified applied fields. In probability distribution theory, it has been realized that classical distributions may not provide a better fit to lifetime data, then a situation arises to introduce an extra parameter to the existing baseline distribution. This extra parameter brings classical distribution into a more reliable and flexible situation while comparing with other distributions. This additional parameter provides superiority and should be significant for modeling data that occur from several applied areas such as engineering, medical sciences, insurance, finance, etc. This additional parameter can be introduced through various techniques. One such technique is the weighted technique. The theory of weighted distribution is useful because it provides a new understanding of existing classical distributions due to the introduction of an additional parameter in the model, which creates flexibility in their nature. Fisher (1934) introduced the concept of weighted distributions to study how the methods of ascertainment can affect the form of the distribution of recorded observations. Later, Rao (1965) developed this concept in a unified manner for problems where the observations fall in non-experimental, non-replicated, and non-random categories. The weighted distributions are used as a tool in the selection of appropriate models for observed data obviously when samples are drawn without a proper frame. The theory of weighted distribution provides integrative conceptualization for model stipulation and data representation problems. The weighted distributions occur especially if observations are recorded by an investigator in nature according to certain stochastic model, the distribution of recorded observation will not have original distribution unless every observation is given an equal chance of being recorded and hence they are recorded according to some weight function. The weighted distributions provide a technique for fitting model to the unknown weight function even if the samples can be taken both from original and developed distributions. The weighted distributions are applied in various research areas related to reliability, biomedicine, ecology, Meta analysis, analysis of family data, analysis of intervention data and other areas for the proper development of statistical models.

There are various authors who illustrated and developed some important weighted probability models along with their statistical features and applications in various fields. Kersey (2010) presented the weighted inverse Weibull distribution and beta-inverse Weibull distribution. Sarma and Das (2021) developed the weighted inverse Nakagami distribution. Mudasir and Ahmad (2017) obtained characterization and information measure of weighted Erlang distribution. Saghir et al. (2017) studies the weighted distributions with brief review, perspective and characterizations. Gharaibeh (2022) developed the weighted Gharaibeh distribution with real data applications. Al-Kadim and Mohammed (2018) introduced the weighted transmuted Pareto distribution. Iqbal and Iqbal (2020) presented the mixiture of weighted exponential and weighted gamma distribution. Reshi and Ahmed (2015) discussed the characterization and estimations of weighted generalized beta probability distribution. Alqallaf et al. (2015) presented weighted exponential distribution and introduce its different methods of estimations. Recently, Ganaie and Rajagopalan (2023) studied the weighted power quasi Lindley distribution with properties and applications of lifetime data.

A generalized Akash distribution is a recently introduced two parametric lifetime distribution developed by Shanker et al. (2018) of which one parameter Akash and exponential distribution are particular cases of it. Its different statistical properties including moments, coefficient of variation, skewness, kurtosis, index of dispersion, mean residual life function, hazard rate function, mean deviation, stochastic ordering, order statistics, Bonferroni and Lorenz curves, Renyi entropy measures and stress-strength reliability has been described and discussed. Further, its parameters have been estimated by using the method of moments and method of maximum likelihood estimation.

2 Weighted Generalized Akash (WGA) Distribution

The probability density function of generalized Akash distribution is given by

$$f(x;\lambda,\beta)=\frac{\lambda^3}{\lambda^2+2\beta}(1+\beta x^2)e^{-\lambda x};\ x>0,\ \lambda>0,\ \beta>0, \tag{1}$$

and the cumulative distribution function of generalized Akash distribution is given by

$$F(x;\lambda,\beta)=1-\left(1+\frac{\beta\lambda x(\lambda x+2)}{\lambda^2+2\beta}\right)e^{-\lambda x};\ x>0,\ \lambda>0,\ \beta>0. \tag{2}$$

Consider X be the random variable following non-negative condition has probability density function . Let its non-negative weight function be , then the probability density function of weighted random variable is given by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))},\quad x>0,$$

where the non-negative weight function be $w(x)$ and $E(w(x)) = \int w(x)f(x)\,dx < \infty$.

In this paper, we have considered the weight function as $w(x) = x^\alpha$ to obtain the weighted version of the generalized Akash distribution, called the weighted generalized Akash distribution. Its probability density function is given by

$$f_w(x) = \frac{x^\alpha f(x)}{E(x^\alpha)}, \tag{3}$$

$$E(x^\alpha) = \int_0^\infty x^\alpha f(x;\lambda,\beta)\,dx,$$

$$E(x^\alpha) = \frac{\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3)}{\lambda^\alpha(\lambda^2+2\beta)}. \tag{4}$$

Now by substituting the equations (1) and (4) in equation (3), we will obtain the required probability density function of weighted generalized Akash distribution as

$$f_w(x) = \frac{x^\alpha\lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(1+\beta x^2)e^{-\lambda x}, \tag{5}$$

and the cumulative distribution function of weighted generalized Akash distribution can be obtained as

$$\begin{aligned} F_w(x) &= \int_0^x f_w(x)\,dx, \\ &= \int_0^x \frac{x^\alpha\lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(1+\beta x^2)e^{-\lambda x}\,dx, \\ &= \frac{1}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\left(\lambda^{\alpha+3}\int_0^x x^\alpha e^{-\lambda x}\,dx + \beta\lambda^{\alpha+3}\int_0^x x^{\alpha+2}e^{-\lambda x}\,dx\right). \end{aligned} \tag{6}$$

Put $\lambda x = t \Rightarrow \lambda\,dx = dt \Rightarrow dx = \frac{dt}{\lambda}$ and $x = \frac{t}{\lambda}$ when $x \to x$, $t \to \lambda x$ and when $x \to 0$, $t \to 0$. After the simplification of equation (6), we will obtain the cumulative distribution function of weighted generalized Akash distribution as

$$F_w(x) = \frac{1}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\left(\lambda^2\gamma(\alpha+1,\lambda x)+\beta\gamma(\alpha+3,\lambda x)\right). \tag{7}$$

3 Survival Analysis

In this section, we will discuss the survival function, hazard rate function, reverse hazard rate function and Mills ratio of the weighted generalized Akash distribution.

The survival function of weighted generalized Akash distribution can be determined as

$$S(x) = 1 - F_w(x) = 1 - \frac{1}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(\lambda^2\gamma(\alpha+1,\lambda x)+\beta\gamma(\alpha+3,\lambda x)).$$

The hazard function is also known as hazard rate or failure rate or force of mortality and is given by

$$h(x) = \frac{f_w(x)}{S(x)} = \frac{x^\alpha\lambda^{\alpha+3}(1+\beta x^2)e^{-\lambda x}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))-(\lambda^2\gamma(\alpha+1,\lambda x)+\beta\gamma(\alpha+3,\lambda x))}.$$

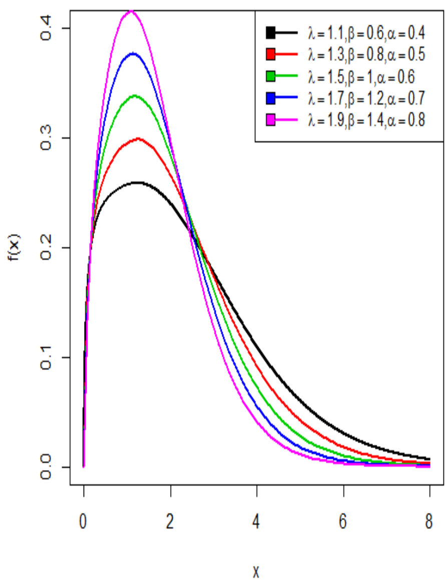

Fig.1:Pdf plot of Weighted Generalized Akash Distribution

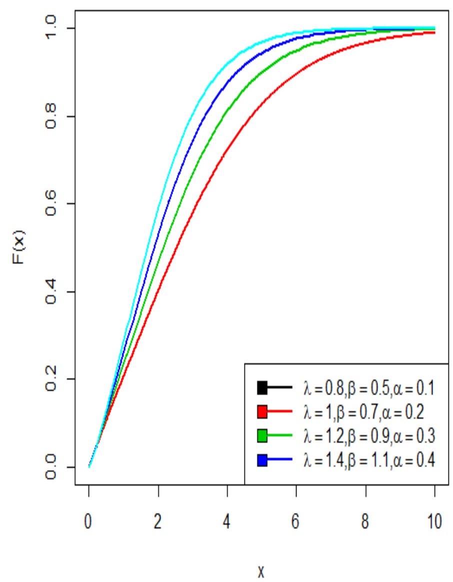

Fig.2:Cdf plot of Weighted Generalized Akash Distribution

The reverse hazard rate function is given by

$$h_r(x) = \frac{f_w(x)}{F_w(x)} = \frac{x^\alpha \lambda^{\alpha+3}(1+\beta x^2)e^{-\lambda x}}{(\lambda^2\gamma(\alpha+1,\lambda x)+\beta\gamma(\alpha+3,\lambda x))}.$$

Mills Ratio is given by

$$M.R = \frac{1}{h_r(x)} = \frac{(\lambda^2\gamma(\alpha+1,\lambda x)+\beta\gamma(\alpha+3,\lambda x))}{x^\alpha \lambda^{\alpha+3}(1+\beta x^2)e^{-\lambda x}}.$$

4 Order Statistics

Consider $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a continuous distribution with probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$. Then, the probability density function of the r-th order statistic $X_{(r)}$ is given by

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)(F_X(x))^{r-1}(1-F_X(x))^{n-r}. \tag{8}$$

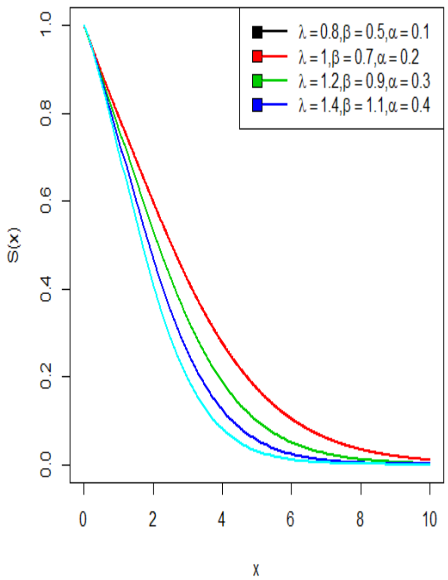

Fig.3:Survival plot of Weighted Generalized Akash Distribution

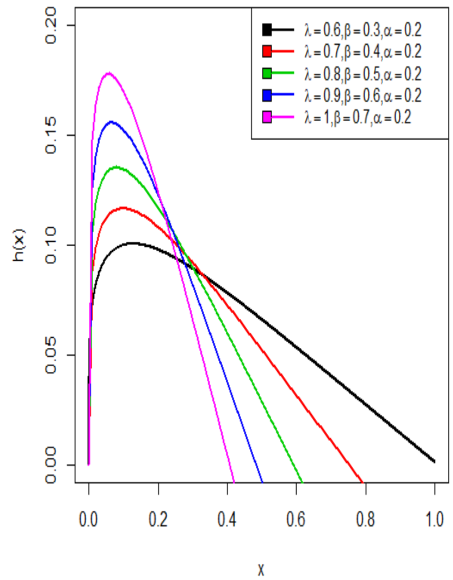

Fig.4:Hazard plot of Weighted Generalized Akash Distribution

Now by substituting the equations (5) and (7) in equation (8), we will obtain the probability density function of r-th order statistics $X_{(r)}$of weighted generalized Akash distribution as

$$\begin{aligned} f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!} & \left(\frac{x^{\alpha}\lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(1+\beta x^2)e^{-\lambda x} \right) \\ & \times \left(\frac{1}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(\lambda^2\gamma(\alpha+1,\lambda x)+\beta\gamma(\alpha+3,\lambda x)) \right)^{r-1} \\ & \times \left(1-\frac{1}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(\lambda^2\gamma(\alpha+1,\lambda x)+\beta\gamma(\alpha+3,\lambda x)) \right)^{n-r}. \end{aligned}$$

Therefore, the probability density function of higher order statistic X(n) of weighted generalized Akash distribution can be determined as

$$\begin{aligned} f_{X(n)}(x) = & \frac{n x^{\alpha}\lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(1+\beta x^2)e^{-\lambda x} \\ & \times \left(\frac{1}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(\lambda^2\gamma(\alpha+1,\lambda x)+\beta\gamma(\alpha+3,\lambda x)) \right)^{n-1}. \end{aligned}$$

and probability density function of first order statistic $X_{(1)}$ of weighted generalized Akash distribution can be determined as

$$f_{X_{(1)}}(x) = \frac{nx^{\alpha}\lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(1+\beta x^2)e^{-\lambda x}$$
$$\times\left(1-\frac{1}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(\lambda^2\gamma(\alpha+1,\lambda x)+\beta\gamma(\alpha+3,\lambda x))\right)^{n-1}.$$

5 Likelihood Ratio Test

Suppose the random sample $X_1, X_2, \ldots, X_n$ of size n from the weighted generalized Akash distribution. To determine its significance the hypothesis is to be analyzed and tested

$$H_0 : f(x) = f(x;\lambda,\beta) \quad \text{against} \quad H_1 : f(x) = f_w(x;\lambda,\beta,\alpha).$$

In order to identify, either if the random sample of size n comes from the generalized Akash distribution or weighted generalized Akash distribution, the given below test statistic procedure is employed.

$$\begin{aligned}\Delta = \frac{L_1}{L_0} &= \prod_{i=1}^{n}\frac{f_w(x;\lambda,\beta,\alpha)}{f(x;\lambda,\beta)},\\ &= \prod_{i=1}^{n}\left(\frac{x_i^{\alpha}\lambda^{\alpha}(\lambda^2+2\beta)}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\right),\\ &= \left(\frac{\lambda^{\alpha}(\lambda^2+2\beta)}{\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3)}\right)^n\prod_{i=1}^{n}x_i^{\alpha}.\end{aligned}$$

We should refuse to accept the null hypothesis, if

$$\Delta = \left(\frac{\lambda^{\alpha}(\lambda^2+2\beta)}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\right)^n\prod_{i=1}^{n}x_i^{\alpha} > k.$$

Equivalently, we should also refuse to retain the null hypothesis where

$$\begin{aligned}\Delta^* &= \prod_{i=1}^{n}x_i^{\alpha} > k\left(\frac{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}{\lambda^{\alpha}(\lambda^2+2\beta)}\right)^n,\\ &= \prod_{i=1}^{n}x_i^{\alpha} > k^*, \text{ where } k^* = k\left(\frac{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}{\lambda^{\alpha}(\lambda^2+2\beta)}\right)^n.\end{aligned}$$

Whether, if the sample is large of size n, $2\log\Delta$ is distributed as a chi-square distribution with one degree of freedom and also the p-value is determined by employing the chi-square distribution. Thus, we should refuse to accept the null hypothesis, if the probability value is given by

$$p(\Delta^* > \theta^*),$$

where $\theta^* = \prod_{i=1}^{n}x_i^{\alpha}$ is lesser than a specified level of significance and $\prod_{i=1}^{n}x_i^{\alpha}$ is the observed value of the statistic Δ^*.

6 Statistical Properties

In this section, we will discuss different structural properties of weighted generalized Akash distribution which include moments, harmonic mean, moment generating function and characteristic function.

Moments: Consider X be a random variable following the weighted generalized Akash distribution. Then the rth order moment $E(X^r)$ of the proposed weighted generalized Akash distribution can be obtained as

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r f_w(x)\,dx, \\ &= \int_0^\infty x^r \left(\frac{x^\alpha \lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(1+\beta x^2)e^{-\lambda x} \right) dx, \\ &= \int_0^\infty \frac{x^{\alpha+r}\lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(1+\beta x^2)e^{-\lambda x}\,dx, \\ &= \frac{\lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\int_0^\infty x^{\alpha+r}(1+\beta x^2)e^{-\lambda x}\,dx, \\ &= \frac{\lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\left(\int_0^\infty x^{(\alpha+r+1)-1}e^{-\lambda x}\,dx + \beta\int_0^\infty x^{(\alpha+r+3)-1}e^{-\lambda x}\,dx \right). \end{aligned} \tag{9}$$

After the simplification of equation (9), we obtain

$$E(X^r) = \mu_r' = \frac{\lambda^2\Gamma(\alpha+r+1)+\beta\Gamma(\alpha+r+3)}{\lambda^r(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}. \tag{10}$$

Now by substituting $r = 1, 2, 3$ and 4 in equation (10), we will obtain the first four moments of weighted generalized Akash distribution as

$$\begin{aligned} E(X) = \mu_1' &= \frac{\lambda^2\Gamma(\alpha+2)+\beta\Gamma(\alpha+4)}{\lambda(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}, \\ E(X^2) = \mu_2' &= \frac{\lambda^2\Gamma(\alpha+3)+\beta\Gamma(\alpha+5)}{\lambda^2(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}, \\ E(X^3) = \mu_3' &= \frac{\lambda^2\Gamma(\alpha+4)+\beta\Gamma(\alpha+6)}{\lambda^3(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}, \\ E(X^4) = \mu_4' &= \frac{\lambda^2\Gamma(\alpha+5)+\beta\Gamma(\alpha+7)}{\lambda^4(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}. \end{aligned}$$

and

$$\begin{aligned} \text{Variance} &= \frac{\lambda^2\Gamma(\alpha+3)+\beta\Gamma(\alpha+5)}{\lambda^2(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))} - \left(\frac{\lambda^2\Gamma(\alpha+2)+\beta\Gamma(\alpha+4)}{\lambda(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))} \right)^2 \\ \text{S.D}(\sigma) &= \sqrt{\frac{\lambda^2\Gamma(\alpha+3)+\beta\Gamma(\alpha+5)}{\lambda^2(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))} - \left(\frac{\lambda^2\Gamma(\alpha+2)+\beta\Gamma(\alpha+4)}{\lambda(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))} \right)^2}. \end{aligned}$$

The harmonic mean of weighted generalized Akash distribution can be obtained as

$$\begin{aligned}
\text{H.M} = E\left(\frac{1}{x}\right) &= \int_0^\infty \frac{1}{x} f_w(x)\,dx, \\
&= \int_0^\infty \frac{1}{x}\frac{x^\alpha \lambda^{\alpha+3}}{\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3)}(1+\beta x^2)e^{-\lambda x}dx, \\
&= \int_0^\infty \frac{x^{\alpha-1}\lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}(1+\beta x^2)e^{-\lambda x}\,dx, \\
&= \frac{\lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\int_0^\infty x^{\alpha-1}(1+\beta x^2)e^{-\lambda x}\,dx, \\
&= \frac{\lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\left(\int_0^\infty x^{(\alpha+1)-2}e^{-\lambda x}\,dx + \beta\int_0^\infty x^{(\alpha+2)-1}e^{-\lambda x}\,dx\right). \qquad (11)
\end{aligned}$$

After the simplification of equation (11), we obtain

$$\text{H.M} = \frac{\lambda\left(\lambda\Gamma(\alpha+1)+\beta\Gamma(\alpha+2)\right)}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}.$$

Moment generating function and characteristic function is given by the random variable X follows a weighted generalized Akash distribution with parameters λ, β, and α. Then the moment generating function of the proposed new distribution can be determined as

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f_w(x)\,dx.$$

Using Taylor's series, we obtain

$$\begin{aligned}
\int_0^\infty e^{tx} f_w(x)\,dx &= \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f_w(x)\,dx = \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!}x^j f_w(x)\,dx \\
&= \sum_{j=0}^\infty \frac{t^j}{j!}\mu_j' = \sum_{j=0}^\infty \frac{t^j}{j!}\left(\frac{\lambda^2\Gamma(\alpha+j+1)+\beta\Gamma(\alpha+j+3)}{\lambda^j(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\right)
\end{aligned}$$

$$M_X(t) = \frac{1}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\sum_{j=0}^\infty \frac{t^j}{j!\lambda^j}\left(\lambda^2\Gamma(\alpha+j+1)+\beta\Gamma(\alpha+j+3)\right).$$

Similarly, the characteristic function of weighted generalized Akash distribution can be obtained as $\varphi_X(t) = M_X(it)$, where

$$M_X(it) = \frac{1}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\sum_{j=0}^\infty \frac{it^j}{j!\lambda^j}\left(\lambda^2\Gamma(\alpha+j+1)+\beta\Gamma(\alpha+j+3)\right).$$

7 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves are also known as classical or income distribution curves which are mostly applied in order to measure the distribution of inequality in income or poverty. The Bonferroni and Lorenz curves are given by

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x f(x)\,dx \ \ and \ \ L(p) = pB(p) = \frac{1}{\mu_1'} \int_0^q x f(x)\,dx,$$

where

$$\begin{aligned}
\mu_1' &= \frac{\lambda^2\Gamma(\alpha+2) + \beta\Gamma(\alpha+4)}{\lambda(\lambda^2\Gamma(\alpha+1) + \beta\Gamma(\alpha+3))} q = F^{-1}(p), \quad and \\
B(p) &= \frac{\lambda(\lambda^2\Gamma(\alpha+1) + \beta\Gamma(\alpha+3))}{p(\lambda^2\Gamma(\alpha+2) + \beta\Gamma(\alpha+4))} \int_0^q x \frac{x^\alpha \lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1) + \beta\Gamma(\alpha+3))} (1+\beta x^2) e^{-\lambda x} dx \\
&= \frac{\lambda(\lambda^2\Gamma(\alpha+1) + \beta\Gamma(\alpha+3))}{p(\lambda^2\Gamma(\alpha+2) + \beta\Gamma(\alpha+4))} \int_0^q \frac{x^{\alpha+1} \lambda^{\alpha+3}}{(\lambda^2\Gamma(\alpha+1) + \beta\Gamma(\alpha+3))} (1+\beta x^2) e^{-\lambda x}\,dx, \\
&= \frac{\lambda^{\alpha+4}}{p(\lambda^2\Gamma(\alpha+2) + \beta\Gamma(\alpha+4))} \int_0^q x^{\alpha+1} (1+\beta x^2) e^{-\lambda x}\,dx, \\
&= \frac{\lambda^{\alpha+4}}{p(\lambda^2\Gamma(\alpha+2) + \beta\Gamma(\alpha+4))} \left(\int_0^q x^{(\alpha+2)-1} e^{-\lambda x}\,dx + \beta \int_0^q x^{(\alpha+4)-1} e^{-\lambda x}\,dx \right).
\end{aligned}$$

After the simplification of above equation, we obtain

$$\begin{aligned}
B(p) &= \frac{\lambda^{\alpha+4}}{p(\lambda^2\Gamma(\alpha+2) + \beta\Gamma(\alpha+4))} \left(\gamma(\alpha+2, \lambda q) + \beta\gamma(\alpha+4, \lambda q) \right) \\
L(p) &= \frac{\lambda^{\alpha+4}}{(\lambda^2\Gamma(\alpha+2) + \beta\Gamma(\alpha+4))} \left(\gamma(\alpha+2, \lambda q) + \beta\gamma(\alpha+4, \lambda q) \right).
\end{aligned}$$

8 Maximum Likelihood Estimation and Fisher's Information Matrix

In this section, we will discuss the method of maximum likelihood estimation to estimate the parameters of the weighted generalized Akash distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the weighted generalized Akash distribution. Then, the likelihood function can be written as

$$\begin{aligned}
L(x) &= \prod_{i=1}^n f_w(x) = \prod_{i=1}^n \left(\frac{x_i^\alpha \lambda^{\alpha+3}}{\lambda^2\Gamma(\alpha+1) + \beta\Gamma(\alpha+3)} (1+\beta x_i^2) e^{-\lambda x_i} \right) \\
L(x) &= \frac{\lambda^{n(\alpha+3)}}{(\lambda^2\Gamma(\alpha+1) + \beta\Gamma(\alpha+3))^n} \prod_{i=1}^n \left(x_i^\alpha (1+\beta x_i^2) e^{-\lambda x_i} \right).
\end{aligned}$$

The log likelihood function is given by

$$\log L = n(\alpha+3)\log\lambda - n\log(\lambda^2\Gamma(\alpha+1) + \beta\Gamma(\alpha+3)) + \alpha\sum_{i=1}^n \log x_i + \sum_{i=1}^n \log(1+\beta x_i^2) - \lambda\sum_{i=1}^n x_i. \quad (12)$$

Now, by differentiating the log likelihood equation (12) with respect to the parameters λ, β, and α, we must satisfy the following normal equations

$$\frac{\partial \log L}{\partial \lambda} = \frac{n(\alpha+3)}{\lambda} - n\left(\frac{(2\lambda\Gamma(\alpha+1))}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\right) - \sum_{i=1}^{n} x_i = 0,$$
$$\frac{\partial \log L}{\partial \beta} = -n\left(\frac{(\Gamma(\alpha+3))}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\right) + \sum_{i=1}^{n}\left(\frac{(x_i^2)}{(1+\beta x_i^2)}\right) = 0,$$
$$\frac{\partial \log L}{\partial \alpha} = n\log\lambda - n\psi(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3)) + \sum_{i=1}^{n}\log x_i = 0,$$

where $\psi(\cdot)$ is the digamma function. The above likelihood equations are too complicated to solve it algebraically. Therefore, we use numerical techniques like the Newton-Raphson method for estimating the required parameters of the proposed distribution.

In order to determine the confidence interval, we use the asymptotic normality results. We have if $\hat{\theta} = (\hat{\lambda}, \hat{\beta}, \hat{\alpha})$ denotes the MLE of $\theta = (\lambda, \beta, \alpha)$. We can obtain the results as

$$\sqrt{n}(\hat{\theta}-\theta) \rightarrow N_3\left(0, \mathcal{I}^{-1}(\theta)\right),$$

where $\mathcal{I}^{-1}(\theta)$ is Fisher's information matrix, i.e.

$$\mathcal{I}(\theta) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \lambda \partial \beta}\right) & E\left(\frac{\partial^2 \log L}{\partial \lambda \partial \alpha}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \beta \partial \lambda}\right) & E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \beta \partial \alpha}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \lambda}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) \end{pmatrix}.$$

where, we can define

$$E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right) = -\frac{n(\alpha+3)}{\lambda^2} - n\left(\frac{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))(2\Gamma(\alpha+1)) - (2\lambda\Gamma(\alpha+1))(2\lambda\Gamma(\alpha+1))}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))^2}\right),$$
$$E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) = n\left(\frac{(\Gamma(\alpha+3))(\Gamma(\alpha+3))}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))^2}\right) - \sum_{i=1}^{n}\left(\frac{E(x_i^2)^2}{(1+\beta x_i^2)^2}\right),$$
$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = -n\psi'(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3)),$$
$$E\left(\frac{\partial^2 \log L}{\partial \lambda \partial \beta}\right) = n\left(\frac{(2\lambda\Gamma(\alpha+1))(\Gamma(\alpha+3))}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))^2}\right),$$
$$E\left(\frac{\partial^2 \log L}{\partial \lambda \partial \alpha}\right) = \frac{n}{\lambda} - n\psi\left(\frac{2\lambda\Gamma(\alpha+1)}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\right),$$
$$E\left(\frac{\partial^2 \log L}{\partial \beta \partial \alpha}\right) = -n\psi\left(\frac{(\Gamma(\alpha+3))}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\right),$$

where $\psi(.)'$ is the first order derivative of the digamma function. Since θ is not known, we estimate $I^{-1}(\theta)$ by $I^{-1}(\hat{\theta})$ and this can be used to obtain asymptotic confidence intervals for λ, β, and α.

9 Application

In this section, we have fitted a real life data set in weighted generalized Akash distribution to discuss its goodness of fit and then comparison has been developed in order to show that the weighted generalized Akash distribution provides a better fit over generalized Akash, quasi Akash and Akash distributions. To compute the model comparison criterion values along with the estimation of unknown parameters, the R software technique is applied to carry out its analysis. In order to compare the performance of the weighted generalized Akash distribution over generalized Akash, quasi Akash, and Akash distributions, we consider criterion values AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), AICC (Akaike Information Criterion Corrected), and $-2\log L$. The distribution is better which shows lesser criterion values of AIC, BIC, AICC, and $-2\log L$. For computing the criterion values like AIC, BIC, AICC, and $-2\log L$, the following formulas are used.

$$\text{AIC} = 2k - 2\log L,$$
$$\text{BIC} = k\log n - 2\log L,$$
$$\text{AICC} = \text{AIC} + \frac{2k(k+1)}{n-k-1},$$

where n is the sample size, k is the number of parameters in the statistical model, and $-2\log L$ is the maximized value of the log-likelihood function under the considered model.

Table 1: Performance of Fitted Distributions

Distributions	**MLE**	**S.E**	**-2logL**	**AIC**	**BIC**	**AICC**
Weighted Generalized Akash	$\hat{\beta} = 0.927$	$\hat{\beta} = 1.427$	180.361	186.361	192.930	186.748
	$\hat{\lambda} = 2.939$	$\hat{\lambda} = 0.508$				
	$\hat{\alpha} = 5.437$	$\hat{\alpha} = 1.531$				
Generalized Akash	$\hat{\beta} = 2391.659$	$\hat{\beta} = 121.885$	204.470	208.470	212.849	208.660
	$\hat{\lambda} = 1.085$	$\hat{\lambda} = 0.000$				
Quasi Akash	$\hat{\beta} = 0.001$	$\hat{\beta} = 0.386$	204.490	208.490	212.869	208.608
	$\hat{\lambda} = 1.085$	$\hat{\lambda} = 0.136$				
Akash	$\hat{\lambda} = 0.883$	$\hat{\lambda} = 0.060$	230.676	232.676	234.865	232.738

From the results given above in Table 1, it has been clearly observed that the weighted generalized Akash distribution has lesser AIC, BIC, AICC, and $-2\log L$ values as compared to the generalized Akash, quasi Akash, and Akash distributions. Hence, it can be concluded that the weighted generalized Akash distribution leads to a better fit than the generalized Akash, quasi Akash, and Akash distributions.

10 Conclusion

In the present paper, we have developed a new distribution called as weighted generalized Akash distribution which has been introduced by using the weighted technique to baseline distribution. Its several statistical features and characteristics such as shape of the behavior of pdf and cdf, moments, harmonic mean, order statistics, survival function, hazard rate function, reverse hazard rate function, moment generating function, Bonferroni and Lorenz curves have been discussed. Furthermore, its parameters have also been estimated through maximum likelihood estimation and also its Fisher's information matrix has been observed. Finally, a real lifetime data set has been analyzed and examined to illustrate the significance of a new distribution and hence it is concluded from the result that weighted generalized Akash distribution provides a quite satisfactory results over generalized Akash, quasi Akash and Akash distributions.

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A Appendix

A.1 R Code for PDF Plot

```
library(zipfR)
rm(list=ls(all=TRUE))
x=seq(0,8,0.000)
y=function(x,lambda=1.1,beta=0.6,alpha=0.4) {((x^alpha)*(lambda^(alpha+3))*(1+beta*x^2)*exp(-lambda*x)) / ((lambda^2)*factorial(alpha)+(beta)*factorial(alpha+2))}
plot(x,y(x),"l",col=1,lwd=2,ylab="f(x)",lty=1,ylim=c(0,0.4),sub="Fig.1: Pdf plot of Weighted Generalized Akash Distribution")
y=function(x,lambda=1.3,beta=0.8,alpha=0.5) {((x^alpha)*(lambda^(alpha+3))*(1+beta*x^2)*exp(-lambda*x)) / ((lambda^2)*factorial(alpha)+(beta)*factorial(alpha+2))}
curve(y,add=T,col=2,lwd=2,lty=1)
y=function(x,lambda=1.5,beta=1.0,alpha=0.6) {((x^alpha)*(lambda^(alpha+3))*(1+beta*x^2)*exp(-lambda*x)) / ((lambda^2)*factorial(alpha)+(beta)*factorial(alpha+2))}
curve(y,add=T,col=3,lwd=2,lty=1)
y=function(x,lambda=1.7,beta=1.2,alpha=0.7) {((x^alpha)*(lambda^(alpha+3))*(1+beta*x^2)*exp(-lambda*x)) / ((lambda^2)*factorial(alpha)+(beta)*factorial(alpha+2))}
curve(y,add=T,col=4,lwd=2,lty=1)
y=function(x,lambda=1.9,beta=1.4,alpha=0.8) {((x^alpha)*(lambda^(alpha+3))*(1+beta*x^2)*exp(-lambda*x)) / ((lambda^2)*factorial(alpha)+(beta)*factorial(alpha+2))}
curve(y,add=T,col=6,lwd=2,lty=1)
legend("topright",legend=c(
  expression(paste(lambda==1.1,",",beta==0.6,",",alpha==0.4)),
  expression(paste(lambda==1.3,",",beta==0.8,",",alpha==0.5)),
  expression(paste(lambda==1.5,",",beta==1.0,",",alpha==0.6)),
  expression(paste(lambda==1.7,",",beta==1.2,",",alpha==0.7)),
  expression(paste(lambda==1.9,",",beta==1.4,",",alpha==0.8))),
  lwd=2,col=c(1,2,3,4,6),text.width=3.0,cex=0.9,fill=c(1,2,3,4,6))
```

A.2 R Code for CDF Plot

```
library(zipfR)
rm(list=ls(all=TRUE))
x=seq(0,10,0.01)
y=function(x,lambda=0.8,beta=0.5,alpha=0.1) {
  ((lambda^2)*Igamma(alpha+1,lambda*x,lower=TRUE)+(beta)*Igamma(alpha+3,
      lambda*x,lower=TRUE)) /
  ((lambda^2)*factorial(alpha)+(beta)*factorial(alpha+2))
}
plot(x,y(x),"l",col=2,lwd=2,ylab="F(x)",lty=1,ylim=c(0,1),sub="Fig.2:Cdf
    plot of Weighted Generalized Akash Distribution")
y=function(x,lambda=1.0,beta=0.7,alpha=0.2) {((lambda^2)*Igamma(alpha+1,
    lambda*x,lower=TRUE)+(beta)*Igamma(alpha+3,lambda*x,lower=TRUE)) / ((
    lambda^2)*factorial(alpha)+(beta)*factorial(alpha+2))}
curve(y,add=T,col=3,lwd=2,lty=1)
y=function(x,lambda=1.2,beta=0.9,alpha=0.3) {((lambda^2)*Igamma(alpha+1,
    lambda*x,lower=TRUE)+(beta)*Igamma(alpha+3,lambda*x,lower=TRUE)) / ((
    lambda^2)*factorial(alpha)+(beta)*factorial(alpha+2))}
curve(y,add=T,col=4,lwd=2,lty=1)
y=function(x,lambda=1.4,beta=1.1,alpha=0.4) {((lambda^2)*Igamma(alpha+1,
    lambda*x,lower=TRUE)+(beta)*Igamma(alpha+3,lambda*x,lower=TRUE)) / ((
    lambda^2)*factorial(alpha)+(beta)*factorial(alpha+2))}
curve(y,add=T,col=5,lwd=2,lty=1)
legend("bottomright",legend=c(
  expression(paste(lambda==0.8,",",beta==0.5,",",alpha==0.1)),
  expression(paste(lambda==1.0,",",beta==0.7,",",alpha==0.2)),
  expression(paste(lambda==1.2,",",beta==0.9,",",alpha==0.3)),
  expression(paste(lambda==1.4,",",beta==1.1,",",alpha==0.4))),
  lwd=2,col=c(1,2,3,4,6),text.width=4.0,cex=1.0,fill=c(1,2,3,4,6))
```

A.3 R Code for Survival Plot

```
library(zipfR)
rm(list=ls(all=TRUE))
x=seq(0,10,0.01)
y=function(x,lambda=0.8,beta=0.5,alpha=0.1) {1-((lambda^2)*Igamma(alpha
    +1,lambda*x,lower=TRUE)+(beta)*Igamma(alpha+3,lambda*x,lower=TRUE))/
    ((lambda^2)*factorial(alpha)+(beta)*factorial(alpha+2))}
plot(x,y(x),"l",col=2,lwd=2,ylab="S(x)",lty=1,ylim=c(0,1),sub="Fig.3:
    Survival plot of Weighted Generalized Akash Distribution")
y=function(x,lambda=1.0,beta=0.7,alpha=0.2) {1-((lambda^2)*Igamma(alpha
    +1,lambda*x,lower=TRUE)+(beta)*Igamma(alpha+3,lambda*x,lower=TRUE))/
    ((lambda^2)*factorial(alpha)+(beta)*factorial(alpha+2))}
curve(y,add=T,col=3,lwd=2,lty=1)
y=function(x,lambda=1.2,beta=0.9,alpha=0.3) { 1-((lambda^2)*Igamma(alpha
    +1,lambda*x,lower=TRUE)+(beta)*Igamma(alpha+3,lambda*x,lower=TRUE))
    /((lambda^2)*factorial(alpha)(beta)*factorial(alpha+2))}
```

```
curve(y,add=T,col=4,lwd=2,lty=1)
y=function(x,lambda=1.4,beta=1.1,alpha=0.4){1((lambda^2)*Igamma(alpha+1,
    lambda*x,lower=TRUE(beta)*Igamma(alpha+3,lambda*x,lower=TRUE))/((
    lambda^2)*factorial(alpha)+(beta)*factorial(alpha+2))}
curve(y,add=T,col=5,lwd=2,lty=1)
legend("topright",legend=c(
  expression(paste(lambda==0.8,",",beta==0.5,",",alpha==0.1)),
  expression(paste(lambda==1.0,",",beta==0.7,",",alpha==0.2)),
  expression(paste(lambda==1.2,",",beta==0.9,",",alpha==0.3)),
  expression(paste(lambda==1.4,",",beta==1.1,",",alpha==0.4))),
  lwd=2,col=c(1,2,3,4,6),text.width=4.0,cex=1.0,fill=c(1,2,3,4,6))
```

A.4 R Code for Hazard Plot

```
library(zipfR)
rm(list=ls(all=TRUE))
x = seq(0, 1, 0.000)
y = function(x, lambda = 0.6, beta = 0.3, alpha = 0.2) {((x^alpha)(lambda
    ^(alpha+3))(1 +betax^2)exp(-lambdax)) /((lambda^2)factorial(alpha) +
    (beta)factorial(alpha+2)) -((lambda^2)Igamma(alpha + 1, lambdax,
    lower = TRUE) +(beta)Igamma(alpha + 3, lambdax, lower= TRUE))}
plot(x, y(x), "l", col = 1, lwd = 2, ylab = "h(x)", lty = 1, ylim = c(0,
    0.20),
sub = "Fig.4: Hazard plot of Weighted Generalized Akash Distribution")
y = function(x, lambda = 0.7, beta = 0.4, alpha = 0.2) {((x^alpha)(lambda
    ^(alpha+3))(1 +betax^2)exp(-lambdax)) /((lambda^2)factorial(alpha) +
    (beta)factorial(alpha+2)) -((lambda^2)Igamma(alpha + 1, lambdax,
    lower = TRUE) +(beta)Igamma(alpha + 3, lambdax, lower = TRUE))}
curve(y, add = TRUE, col = 2, lwd = 2, lty = 1)
y = function(x, lambda = 0.8, beta = 0.5, alpha = 0.2) {((x^alpha)(lambda
    ^(alpha+3))(1 +betax^2)exp(-lambdax)) /((lambda^2)factorial(alpha) +
    (beta)factorial(alpha+2)) -((lambda^2)Igamma(alpha + 1, lambdax,
    lower = TRUE) +(beta)Igamma(alpha + 3, lambdax, lower = TRUE))}
curve(y, add = TRUE, col = 3, lwd = 2, lty = 1)
y = function(x, lambda = 0.9, beta = 0.6, alpha = 0.2) {((x^alpha)(lambda
    ^(alpha+3))(1 +betax^2)exp(-lambdax)) /((lambda^2)factorial(alpha) +
    (beta)factorial(alpha+2)) -((lambda^2)Igamma(alpha + 1, lambdax,
    lower = TRUE) +(beta)Igamma(alpha + 3, lambdax, lower= TRUE))}
curve(y, add = TRUE, col = 4, lwd = 2, lty = 1)
y = function(x, lambda = 1.0, beta = 0.7, alpha = 0.2) {((x^alpha)(lambda
    ^(alpha+3))(1 + beta*x^2)exp(-lambdax)) /((lambda^2)*factorial(alpha)
     + (beta)*factorial(alpha+2)) -((lambda^2)Igamma(alpha + 1, lambdax,
    lower = TRUE) +(beta)Igamma(alpha + 3, lambdax, lower= TRUE))}
curve(y, add = TRUE, col = 6, lwd = 2, lty = 1)
legend("topright", legend = c(
expression(paste(lambda == 0.6, ",", beta == 0.3, ",", alpha == 0.2)),
expression(paste(lambda == 0.7, ",", beta == 0.4, ",", alpha == 0.2)),
expression(paste(lambda == 0.8, ",", beta == 0.5, ",", alpha == 0.2)),
```

```
expression(paste(lambda == 0.9, ",", beta == 0.6, ",", alpha == 0.2)),
expression(paste(lambda == 1.0, ",", beta == 0.7, ",", alpha == 0.2))
), lwd = 2, col = c(1, 2, 3, 4, 6), text.width = 0.4, cex = 0.9, fill = c
    (1, 2, 3, 4, 6))
```

B R Code for Application

B.1 Weighted Generalized Akash Distribution

```
strength_data = c(0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69,
    1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43,
    2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79,
    2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11,
    3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33,
    3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42,
    4.70, 4.90)
dburr = function(x, alpha, lambda, beta) {(x^alpha * (lambda^(alpha+3)) *
     (1 + betax^2)exp(-lambdax)) /(lambda^2 * factorial(alpha) + (beta) *
     factorial(alpha + 2))}
library(MASS)
MLE_lambda_beta_alpha = fitdistr(x = strength_data, densfun = dburr,start
     = list(beta = 0.5, lambda = 0.91, alpha = 1.3),lower = c(0.001,
    0.001, 0.001), upper = c(Inf, Inf, Inf))
MLE_lambda_beta_alpha
Minus_2_logL1 = -2 * fitdistr(x = strength_data, densfun = dburr,start =
    list(beta = 0.5, lambda = 0.91, alpha = 1.3),lower = c(0.001, 0.001,
    0.001), upper = c(Inf, Inf, Inf))$loglik
Minus_2_logL1
AIC(MLE_lambda_beta_alpha)
BIC(MLE_lambda_beta_alpha)
dburr = function(x, beta = 5.478, lambda = 7.489, alpha = 3.467) {(x^
    alpha (lambda^(alpha+3)) * (1 + betax^2) * exp(-lambdax)) /(lambda^2
    * factorial(alpha) + (beta) * factorial(alpha + 2))}
curve(dburr(x), add = TRUE, col = 3, lwd = 2)
legend("topright", legend = c("Weighted Generalized Akash Distribution","
    Generalized Akash Distribution","Quasi Akash Distribution","Akash
    Distribution"),pt.cex = 13, cex = 0.9, lty = 2:3, fill = 2:3, lwd =
    2)
strength_data=c(0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69,
    1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43,
    2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79,
    2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11,
    3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33,
    3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42,
    4.70, 4.90)
```

B.2 Generalized Akash Distribution

```
dburr = function(x, lambda, beta) {((lambda^3) * (1 + betax^2) * exp(-lambdax)) /((lambda^2) + 2 * beta)}
library(MASS)
MLE_lambda_beta = fitdistr(x = strength_data, densfun = dburr,
start = list(beta = 0.5, lambda = 0.91),
lower = c(0.001, 0.001), upper = c(Inf, Inf))
MLE_lambda_beta
Minus_2_logL = -2 * fitdistr(x = strength_data, densfun = dburr,start = list(beta = 0.5,lambda = 0.91),
lower = c(0.001, 0.001), upper = c(Inf, Inf))$loglik
Minus_2_logL
AIC(MLE_lambda_beta)
BIC(MLE_lambda_beta)
hist(strength_data, probability = TRUE, breaks = 20, col = "skyblue", main = "Histogram of Strength Data",xlab = "Strength", ylab = "Density")
curve(dburr(x, lambda = MLE_lambda_beta$estimate[1], beta = MLE_lambda_beta$estimate[2]),add = TRUE, col = "red", lwd = 2)
strength_data=c(0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80,
1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50,
2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85,
2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19,
3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65,
3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90)
```

B.3 Quasi Akash Distribution

```
dburr=function(x,lambda,beta) (lambda^2*(beta+lambda*x^2)*exp(-lambda*x))/(beta*lambda+2)
library(MASS)
MLE_lambda_beta=fitdistr(x = strength_data,densfun = dburr,start = list(beta=0.5,lambda=0.91),lower=c(0.001,0.001),upper=c(Inf,Inf))
MLE_lambda_beta
Minus_2_logL=-2*fitdistr(x = strength_data,densfun = dburr,start = list(beta=0.5,lambda=0.91),lower=c(0.001,0.001),upper=c(Inf,Inf))$loglik
Minus_2_logL
AIC(MLE_lambda_beta)
BIC(MLE_lambda_beta)
hist(strength_data,prob=TRUE,density=c(15),main="Fig.5:Fitting of W.G. Akash Distt, G.Akash Distt, Quasi Akash and Akash distribution to the relief time data given in table.1",xlab= "observed")
dburr=function(x,beta=5.478,lambda=7.489) (lambda^2*(beta+lambda*x^2)*exp(-lambda*x))/(beta*lambda+2)
curve(dburr(x),add=TRUE,col=3,lwd=2)
```

B.4 Akash Distribution

```
dburr=function(x,lambda) (lambda^3*(1+x^2)*exp(-lambda*x))/(lambda^2+2)
library(MASS)
MLE_lambda=fitdistr(x = strength_data,densfun = dburr,start = list(lambda=0.91),lower=c(0.001),upper=c(Inf))
MLE_lambda
Minus_2_logL=-2*fitdistr(x = strength_data,densfun = dburr,start = list(lambda=0.91),lower=c(0.001),upper=c(Inf))$loglik
Minus_2_logL
AIC(MLE_lambda)
BIC(MLE_lambda)
hist(strength_data,prob=TRUE,density=c(15),main="Fig.5:Fitting of W.G. Akash Distt, G.Akash Distt, Quasi Akash and Akash distribution to the relief time data given in table.1",xlab= "observed")
dburr=function(x,lambda=7.489) (lambda^3*(1+x^2)*exp(-lambda*x))/(lambda^2+2)
curve(dburr(x),add=TRUE,col=3,lwd=2)
```

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