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A NEW EXTENSION OF GENERALIZED ARADHANA DISTRIBUTION WITH PROPERTIES AND ITS APPLICATIONS USING R PROGRAMMING

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SUMMARY

In this paper, we introduce a new class of generalized Aradhana distribution with suitable R code termed as area biased generalized Aradhana distribution. The proposed new distribution is a special case of broader weighted distribution family. Its several structural properties have been explored thoroughly and furthermore its parameters have also been estimated based on maximum likelihood estimation. To assess its supremacy, a real lifetime data set has been fitted in proposed new distribution to determine its flexibility and superiority in comparison to existing classical distribution.

Keywords: : Generalized Aradhana Distribution, Weighted Distribution, Order Statistics, Reliability Measures, Maximum Likelihood Estimation.

1 Introduction

The theory of weighted distribution has retained a prominent place because it provides a significant role in handling different lifetime data sets occurring from various applied fields like engineering, medical sciences, finance, insurance etc. The idea of weighted distribution was proposed by Fisher (1934) in connection with his studies on how the methods of ascertainment can influence the form of distribution of recorded observations. Later, Rao (1965) developed this concept in a unified manner in association with modeling statistical data were the routine practice of employing classical distributions for the purpose was found to be inappropriate. The weighted distribution gives a method for fitting model to unknown weight function when samples can be taken both from original and developed distribution. The weighted distributions are remarkable for efficient modeling of statistical data and prediction obviously when existing distributions are not appropriate. The weighted distribution reduces to length biased distribution when the weight function considers only length of the units of interest.The concept of length biased sampling was introduced by Cox (1969) and Zelen (1974). The statistical interpretation of length biased distribution was originally introduced by Cox (1962) in renewal theory.The length biased and area biased distributions are special cases of weighted distributions and hence may be defined as when sample observations have unequal probability of selection, then we apply weights to the distribution to model the bias. The area biased distributions have been employed largely for sampling in forestry, medical sciences, psychology etc. Length biased sampling situation occurs in clinical trials, reliability theory and population studies were a proper sampling frame is absent.

There are various authors who developed some important area biased weighted probability models along with their illustrations and applications in various fields. Bashir and Mahmood (2019) developed multivariate area biased Lindley distribution with properties and applications. Eyob et al. (2019) presented weighted quasi Akash distribution and obtain its properties and applications. Bashir and Rasul (2018) discussed on the area biased Rayleigh distribution with properties and applications on lifetime data. Beghriche and Zeghdoudi (2019) obtained size biased gamma Lindley distribution.Aijaz et al. (2022) introduced poisson area biased Ailamujia distribution with its applications in environmental and medical sciences. Reyadet al. (2017) proposed length biased weighted frechet distribution with properties and estimation.Elangovan and Mohanasundari (2019) presented the area biased Aradhana distribution with applications in cancer data. Ade et al.(2020) proposed area biased generalized uniform distribution with some statistical properties. Perveen et al. (2016) presented area biased weighted weibull distribution. Sharma et al. (2018) discussed on the length and area biased Maxwell distribution.Fazal (2018) obtained the area-biased poisson exponential distribution with applications. Bashir and Rasul (2016) presented poisson area-biased Lindley distribution with applications on biological data.

A generalized Aradhana distribution is two parametric continuous lifetime distribution introduced by Welday and Shanker (2018) which is a special case of one parameter exponential and Aradhana distribution. Its different statistical properties like hazard rate function, mean residual life function, shapes of pdf for varying values of parameters, stochastic ordering, mean deviations, stress-strength reliability, coefficient of variation, skewness, kurtosis, index of dispersion, bonferroni and lorenz curves have been discussed. Its parameters have also been estimated by using the maximum likelihood estimation.

2 Area Biased Generalized Aradhana Distribution

The probability density function of generalized Aradhana distribution is given by

$$f(x;\theta,\alpha) = \frac{\theta^3}{\theta^2 + 2\alpha\theta + 2\alpha^2}(1+\alpha x)^2 e^{-\theta x};\ x>0,\ \theta>0,\ \alpha\geq 0, \tag{1}$$

and the cumulative distribution function of generalized Aradhana distribution is given by

$$F(x;\theta,\alpha) = 1 - \left(1 + \frac{\alpha\theta x(2\theta + \alpha\theta x + 2\alpha)}{\theta^2 + 2\alpha\theta + 2\alpha^2}\right) e^{-\theta x};\ x>0,\ \theta>0,\ \alpha\geq 0. \tag{2}$$

Let X be a random variable representing a non-negative condition with probability density function $f_X(x)$. Let its non-negative weight function be $w(x)$. Then the probability density function of the weighted random variable X_w is given by

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))}, \quad x>0,$$

where $w(x)$ is a non-negative weight function and $E(w(x)) = \int w(x)f(x)\,dx < \infty$.

Depending upon various forms of weight function $w(x)$, specifically when $w(x) = x^c$,the resulting distribution is termed as weighted distribution. In this paper, we study area biased version of generalized Aradhana distribution known as area biased generalized Aradhana distribution.For the weight function $w(x) = x^2$, the resulting distribution is termed as area biased distribution with probability density function given by

$$f_a(x) = \frac{x^2 f(x)}{E(x^2)}, \tag{3}$$

where

$$E(x^2) = \int_0^\infty x^2 f(x;\theta,\alpha)\,dx = \frac{(2\theta^2 + 24\alpha^2 + 12\alpha\theta)}{\theta^2(\theta^2 + 2\alpha\theta + 2\alpha^2)}. \tag{4}$$

Now using equations (1) and (4) in equation (3), we get the required probability density function of area biased generalized Aradhana distribution for $\alpha = \beta$ and $\theta = \lambda$,

$$f_\beta(x) = \frac{x^2\lambda^5}{(2\lambda^2 + 24\beta^2 + 12\beta\lambda)}(1+\beta x)^2 e^{-\lambda x}. \tag{5}$$

The cumulative distribution function of area biased generalized Aradhana distribution can be determined as

$$\begin{aligned} F(x) &= \int_0^x f_\beta(x)\,dx = \int_0^x \frac{x^2\lambda^5}{(2\lambda^2 + 24\beta^2 + 12\beta\lambda)}(1+\beta x)^2 e^{-\lambda x}\,dx \\ &= \frac{1}{(2\lambda^2 + 24\beta^2 + 12\beta\lambda)}\int_0^x x^2\beta^5(1+\beta x)^2 e^{-\lambda x}\,dx \\ &= \frac{\lambda^5}{(2\lambda^2 + 24\beta^2 + 12\beta\lambda)}\left(\int_0^x x^2 e^{-\lambda x}\,dx + \beta^2\int_0^x x^4 e^{-\lambda x}\,dx + 2\beta\int_0^x x^3 e^{-\lambda x}\,dx\right). \end{aligned} \tag{6}$$

After simplification of equation (6), we get the cumulative distribution function of area biased generalized Aradhana distribution as

$$F_\beta(x) = \frac{1}{(2\lambda^2 + 24\beta^2 + 12\beta\lambda)}\left(\lambda^2\Gamma(3,\lambda x) + \beta^2\Gamma(5,\lambda x) + 2\beta\lambda\Gamma(4,\lambda x)\right). \tag{7}$$

3 Reliability Measures

In this section, we will derive the reliability function, hazard rate function, reverse hazard rate function and Mills ratio of the developed area biased generalized Aradhana distribution.

The reliability function, also termed as survival function, of area biased generalized Aradhana distribution can be computed as

$$R(x) = 1 - F_a(x) = 1 - \frac{1}{(2\lambda^2 + 24\beta^2 + 12\beta\lambda)}\left(\lambda^2\Gamma(3,\lambda x) + \beta^2\Gamma(5,\lambda x) + 2\beta\lambda\Gamma(4,\lambda x)\right).$$

The hazard function, also known as hazard rate or failure rate or force of mortality, is given by

$$h(x) = \frac{f_\beta(x)}{R(x)} = \frac{x^2\lambda^5(1+\beta x)^2 e^{-\lambda x}}{(2\lambda^2 + 24\beta^2 + 12\beta\lambda) - (\lambda^2\Gamma(3,\lambda x) + \beta^2\Gamma(5,\lambda x) + 2\beta\lambda\Gamma(4,\lambda x))}.$$

The reverse hazard rate function is given by

$$h_r(x) = \frac{f_\beta(x)}{F_\beta(x)} = \frac{x^2\lambda^5(1+\beta x)^2 e^{-\lambda x}}{\lambda^2\Gamma(3,\lambda x) + \beta^2\Gamma(5,\lambda x) + 2\beta\lambda\Gamma(4,\lambda x)}.$$

The Mills Ratio is given by

$$M.R. = \frac{1}{h_r(x)} = \frac{(\lambda^2\gamma(3,\lambda x) + \beta^2\gamma(5,\lambda x) + 2\beta\lambda\gamma(4,\lambda x))}{x^2\lambda^5(1+\beta x)^2 e^{-\lambda x}}.$$

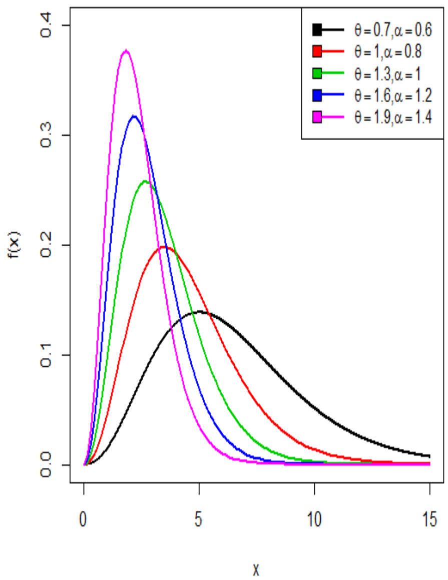

Fig.1:Pdf plot of Area Biased Generalized Aradhana Distribution

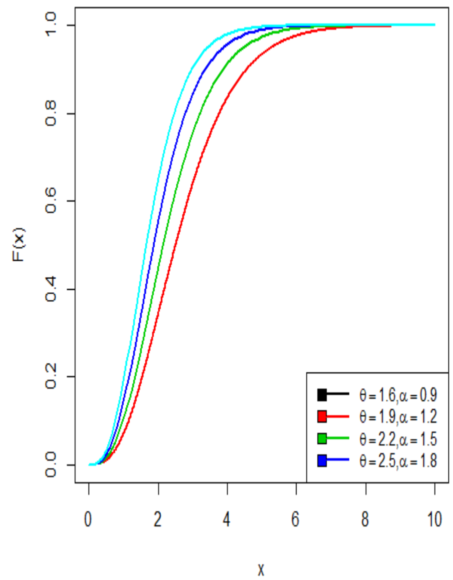

Fig.2:Cdf plot of Area Biased Generalized Aradhana Distribution

4 Order Statistics

Order statistics play a key role in statistical sciences and have a wide range of applications in the fields of modeling auctions, car races, insurance, finance, etc. Consider $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ to be the order statistics of a random sample $X_1, X_2, \dots, X_n$ from a continuous population with probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$. Then the probability density function of the r-th order statistic $X_{(r)}$ is given by

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)(F_X(x)])^{r-1}(1-F_X(x))^{n-r}. \tag{8}$$

Now, by substituting the equations (5) and (7) into equation (8), we obtain the required probability density function of the r-th order statistic $X_{(r)}$ of the weighted generalized Aradhana distribution as

$$\begin{aligned} f_{x(r)}(x) = {} & \frac{n!}{(r-1)!(n-r)!} \left(\frac{x^2\lambda^5}{(2\lambda^2+24\beta^2+12\beta\lambda)} (1+\beta x)^2 e^{-\lambda x} \right) \\ & \times \left(\frac{1}{(2\lambda^2+24\beta^2+12\beta\lambda)} \left(\lambda^2\gamma(3,\lambda x) + \beta^2\gamma(5,\lambda x) + 2\beta\lambda\gamma(4,\lambda x) \right) \right)^{r-1} \\ & \times \left(1 - \frac{1}{(2\lambda^2+24\beta^2+12\beta\lambda)} \left(\lambda^2\gamma(3,\lambda x) + \beta^2\gamma(5,\lambda x) + 2\beta\lambda\gamma(4,\lambda x) \right) \right)^{n-r}. \end{aligned}$$

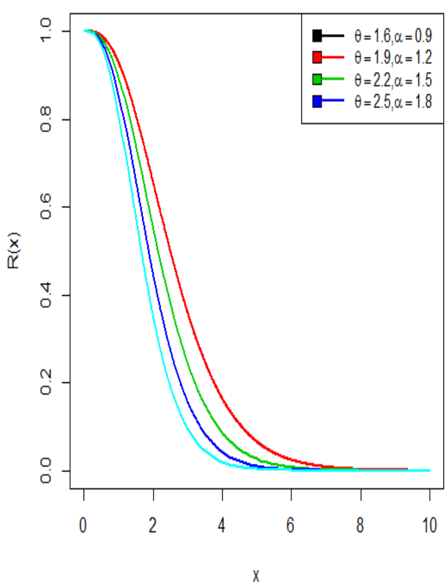

Fig.3:Reliability plot of Area Biased Generalized Aradhana Distribution

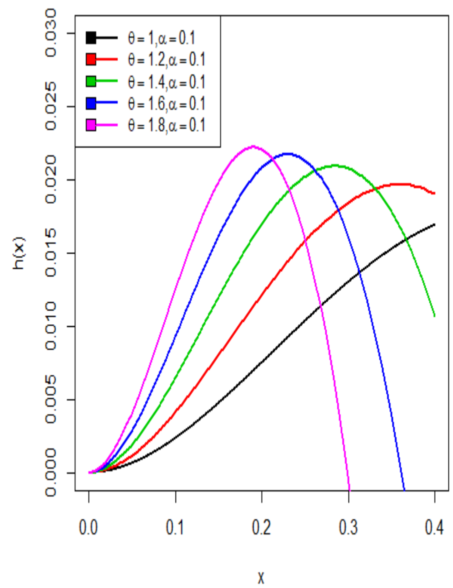

Fig.4:Hazard plot of Area Biased Generalized Aradhana Distribution

Therefore, the probability density function of higher order statistic $X_{(n)}$ of area biased generalized Aradhana distribution can be determined as

$$\begin{aligned} f_{x(n)}(x) = {} & \frac{nx^2\lambda^5}{(2\lambda^2+24\beta^2+12\beta\lambda)}(1+\beta x)^2 e^{-\lambda x} \\ & \times \left(\frac{1}{(2\lambda^2+24\beta^2+12\beta\lambda)} \left(\lambda^2\gamma(3,\lambda x)+\beta^2\gamma(5,\lambda x)+2\beta\lambda\gamma(4,\lambda x)\right)\right)^{n-1}, \end{aligned}$$

and the probability density function of first order statistic $X_{(1)}$ of area biased generalized Aradhana distribution can be determined as

$$\begin{aligned} f_{x(1)}(x) = {} & \frac{nx^2\lambda^5}{(2\lambda^2+24\beta^2+12\beta\lambda)}(1+\beta x)^2 e^{-\lambda x} \\ & \times \left(1 - \frac{1}{(2\lambda^2+24\beta^2+12\beta\lambda)} \left(\lambda^2\gamma(3,\lambda x)+\beta^2\gamma(5,\lambda x)+2\beta\lambda\gamma(4,\lambda x)\right)\right)^{n-1}. \end{aligned}$$

5 Likelihood Ratio Test

Suppose we have the random sample $X_1, X_2, \ldots, X_n$ of size n drawn from the generalized Aradhana or area biased generalized Aradhana distribution. To analyze its significance, we consider testing the hypothesis

$$H_0 : f(x) = f(x;\lambda,\beta) \quad \text{against} \quad H_1 : f(x) = f_\beta(x;\lambda,\beta).$$

To determine whether the random sample of size n comes from the generalized Aradhana distribution or area biased generalized Aradhana distribution, the below test statistic procedure is used.

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n} \frac{f_\beta(x;\lambda,\beta)}{f(x;\lambda,\beta)} = \prod_{i=1}^{n} \left(\frac{x_i^2\lambda^2\left(\lambda^2+2\beta\lambda+2\beta^2\right)}{(2\lambda^2+24\beta^2+12\beta\lambda)} \right) = \left(\frac{\lambda^2\left(\lambda^2+2\beta\lambda+2\beta^2\right)}{(2\lambda^2+24\beta^2+12\beta\lambda)} \right)^n \prod_{i=1}^{n} {x_i}^2.$$

We should refuse to retain the null hypothesis if

$$\Delta = \left(\frac{\lambda^2\left(\lambda^2+2\beta\lambda+2\beta^2\right)}{(2\lambda^2+24\beta^2+12\beta\lambda)} \right)^n \prod_{i=1}^{n} x_i^2 > k.$$

Equivalently, we should reject the null hypothesis if

$$\Delta^* = \prod_{i=1}^{n} x_i^2 > k \left(\frac{\lambda^2\left(\lambda^2+2\beta\lambda+2\beta^2\right)}{(2\lambda^2+24\beta^2+12\beta\lambda)} \right)^{-n},$$

or

$$\Delta^* = \prod_{i=1}^{n} x_i^2 > k^*, \text{ where } k^* = k \left(\frac{\lambda^2\left(\lambda^2+2\beta\lambda+2\beta^2\right)}{(2\lambda^2+24\beta^2+12\beta\lambda)} \right)^{-n}.$$

As $2\log\Delta$ is distributed as chi-square with one degree of freedom if the sample size n is large, the p-value is determined by employing the chi-square distribution. Thus we should reject the null hypothesis if the probability $p\left(\Delta^* > \lambda^*\right)$, where $\lambda^* = \prod_{i=1}^{n} {x_i}^2$, is lower than a given level of significance.

6 Statistical Properties

In this section, we discuss different statistical properties of area biased generalized Aradhana distribution, which include moments, harmonic mean, moment generating function and characteristic function.

6.1 Moments

Let X be the random variable following area biased generalized Aradhana distribution with parameters θ and α. Then the r^{th} order moment of the distribution can be determined as

$$\begin{aligned}
E\left(X^r\right) = \mu_r' &= \int_0^\infty x^r f_a(x)\,dx = \int_0^\infty x^r \frac{x^2\lambda^5}{(2\lambda^2+24\beta^2+12\beta\lambda)}(1+\beta x)^2 e^{-\lambda x}\,dx \\
&= \int_0^\infty \frac{x^{r+2}\lambda^5}{(2\lambda^2+24\beta^2+12\beta\lambda)}(1+\beta x)^2 e^{-\lambda x}\,dx \\
&= \frac{\lambda^5}{(2\lambda^2+24\beta^2+12\beta\lambda)} \int_0^\infty x^{r+2}(1+\beta x)^2 e^{-\lambda x}\,dx \\
&= \frac{\lambda^5}{(2\lambda^2+24\beta^2+12\beta\lambda)} \int_0^\infty x^{r+2}\left(1+\beta^2x^2+2\beta x\right) e^{-\lambda x}\,dx \\
&= \frac{\lambda^5}{(2\lambda^2+24\beta^2+12\beta\lambda)} \\
&\times \left(\int_0^\infty x^{(r+3)-1} e^{-\lambda x}\,dx + \alpha^2 \int_0^\infty x^{(r+5)-1} e^{-\lambda x}\,dx + 2\beta \int_0^\infty x^{(r+4)-1} e^{-\lambda x}\,dx \right). \quad (9)
\end{aligned}$$

After simplification, equation (9) gives

$$E\left(X^r\right)=\mu_r'=\frac{\lambda^2\Gamma(r+3)+\lambda^2\Gamma(r+5)+2\beta\lambda\Gamma(r+4)}{\lambda^r\left(2\lambda^2+24\beta^2+12\beta\lambda\right)}. \tag{10}$$

Now substituting $r=1,2,3$ and 4 in equation (10), we get the first four moments of the area biased generalized Aradhana distribution as

$$\begin{aligned}
E(X)=\mu_1'&=\frac{6\lambda^2+120\beta^2+48\beta\lambda}{\lambda\left(2\lambda^2+24\beta^2+12\beta\lambda\right)},\\
E\left(X^2\right)=\mu_2'&=\frac{24\lambda^2+720\beta^2+240\beta\lambda}{\lambda^2\left(2\lambda^2+24\beta^2+12\beta\lambda\right)},\\
E\left(X^3\right)=\mu_3'&=\frac{120\lambda^2+5040\beta^2+1440\beta\lambda}{\theta^3\left(2\lambda^2+24\beta^2+12\beta\lambda\right)}, \quad \text{and}\\
E\left(X^4\right)=\mu_4'&=\frac{720\lambda^2+40320\beta^2+10080\beta\lambda}{\lambda^4\left(2\lambda^2+24\beta^2+12\beta\lambda\right)}.
\end{aligned}$$

Then the Variance,

$$\mathrm{V}=\frac{24\lambda^2+720\beta^2+240\beta\lambda}{\lambda^2\left(2\lambda^2+24\beta^2+12\beta\lambda\right)}-\left(\frac{6\lambda^2+120\beta^2+48\beta\lambda}{\lambda\left(2\lambda^2+24\beta^2+12\beta\lambda\right)}\right)^2,$$

and Standard Deviation,

$$S.D=\sqrt{\frac{24\lambda^2+720\beta^2+240\beta\lambda}{\lambda^2\left(2\lambda^2+24\beta^2+12\beta\lambda\right)}-\left(\frac{6\lambda^2+120\beta^2+48\beta\lambda}{\lambda\left(2\lambda^2+24\beta^2+12\beta\lambda\right)}\right)^2}.$$

6.2 Harmonic mean

The harmonic mean for developed area biased generalized Aradhana distribution can be determined as

$$\begin{aligned}
H.M.=E(1/x)&=\int_0^\infty \frac{1}{x}f_a(x)\,dx=\int_0^\infty \frac{1}{x}\frac{x^2\lambda^5}{\left(2\lambda^2+24\beta^2+12\beta\lambda\right)}(1+\beta x)^2e^{-\lambda x}\,dx\\
&=\int_0^\infty \frac{x\lambda^5}{\left(2\lambda^2+24\beta^2+12\beta\lambda\right)}\left(1+\beta^2x^2+2\beta x\right)e^{-\lambda x}\,dx\\
&=\frac{\lambda^5}{\left(2\lambda^2+24\beta^2+12\beta\lambda\right)}\int_0^\infty x\left(1+\beta^2x^2+2\beta x\right)e^{-\lambda x}\,dx\\
&=\frac{\lambda^5}{\left(2\lambda^2+24\beta^2+12\beta\lambda\right)}\\
&\times\left(\int_0^\infty x^{(3)-2}e^{-\lambda x}\,dx+\beta^2\int_0^\infty x^{(4)-1}e^{-\lambda x}\,dx+2\beta\int_0^\infty x^{(3)-1}e^{-\lambda x}\,dx\right).
\end{aligned} \tag{11}$$

After simplification, equation (11) gives

$$H.M.=\frac{\lambda\left(2\lambda+6\beta^2+4\beta\lambda\right)}{\left(2\beta^2+24\beta^2+12\beta\lambda\right)}.$$

6.3 Moment generating function and characteristic function

Suppose the random variable X follows a weighted generalized Akash distribution with parameters λ, β, and α. Then the moment generating function of the proposed new distribution can be determined as

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f_w(x)\,dx.$$

Using Taylor's series, we obtain

$$\begin{aligned} M_X(t) &= \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f_w(x)\,dx = \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f_w(x)\,dx = \sum_{j=0}^\infty \frac{t^j}{j!}\mu_j' \\ &= \sum_{j=0}^\infty \frac{t^j}{j!}\left(\frac{\lambda^2\Gamma(\alpha+j+1)+\beta\Gamma(\alpha+j+3)}{\lambda^j(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\right) \\ &= \frac{1}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\sum_{j=0}^\infty \frac{t^j}{j!\lambda^j}\left(\lambda^2\Gamma(\alpha+j+1)+\beta\Gamma(\alpha+j+3)\right). \end{aligned}$$

Similarly, the characteristic function of the weighted generalized Akash distribution can be obtained as

$$\varphi_X(t) = M_X(it) = \frac{1}{(\lambda^2\Gamma(\alpha+1)+\beta\Gamma(\alpha+3))}\sum_{j=0}^\infty \frac{it^j}{j!\lambda^j}\left(\lambda^2\Gamma(\alpha+j+1)+\beta\Gamma(\alpha+j+3)\right).$$

7 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves, also termed as classical curves, are mostly employed to study the distribution of inequality in income or poverty. The Bonferroni and Lorenz curves can be defined as

$$B(p) = \frac{1}{p\mu_1'}\int_0^q x f(x)\,dx \quad \text{and} \quad L(p) = pB(p) = \frac{1}{\mu_1'}\int_0^q x f(x)\,dx,$$

where $\mu_1' = \frac{6\theta^2+120\alpha^2+48\alpha\theta}{\theta\left(2\theta^2+24\alpha^2+12\alpha\theta\right)}$ and $q = F^{-1}(p)$. After some algebra

$$\begin{aligned} B(p) &= \frac{\theta\left(2\theta^2+24\alpha^2+12\alpha\theta\right)}{p\left(6\theta^2+120\alpha^2+48\alpha\theta\right)}\int_0^q x\frac{x^2\theta^5}{(2\theta^2+24\alpha^2+12\alpha\theta)}(1+\alpha x)^2 e^{-\theta x}\,dx \\ &= \frac{\theta\left(2\theta^2+24\alpha^2+12\alpha\theta\right)}{p\left(6\theta^2+120\alpha^2+48\alpha\theta\right)}\int_0^q \frac{x^3\theta^5}{(2\theta^2+24\alpha^2+12\alpha\theta)}\left(1+\alpha^2x^2+2\alpha x\right)e^{-\theta x}\,dx \\ &= \frac{\theta^6}{p\left(6\theta^2+120\alpha^2+48\alpha\theta\right)}\int_0^q x^3\left(1+\alpha^2x^2+2\alpha x\right)e^{-\theta x}\,dx \\ &= \frac{\theta^6}{p\left(6\theta^2+120\alpha^2+48\alpha\theta\right)}\left(\int_0^q x^{(4)-1}e^{-\theta x}\,dx + \alpha^2\int_0^q x^{(6)-1}e^{-\theta x}\,dx + 2\alpha\int_0^q x^{(5)-1}e^{-\theta x}\,dx\right). \end{aligned}$$

After simplification, we obtain

$$\begin{aligned} B(p) &= \frac{\theta^6}{p\left(6\theta^2+120\alpha^2+48\alpha\theta\right)}\left(\gamma(4,\theta q)+\alpha^2\gamma(6,\theta q)+2\alpha\gamma(5,\theta q)\right) \\ L(p) &= \frac{\theta^6}{\left(6\theta^2+120\alpha^2+48\alpha\theta\right)}\left(\gamma(4,\theta q)+\alpha^2\gamma(6,\theta q)+2\alpha\gamma(5,\theta q)\right). \end{aligned}$$

8 Maximum Likelihood Estimation and Fisher's Information Matrix

In this section, we discuss the maximum likelihood method to estimate the parameters of the area-biased generalized Aradhana distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the area-biased generalized Aradhana distribution. Then the likelihood function is obtained as

$$L(x) = \prod_{i=1}^{n} f_\beta(x) = \prod_{i=1}^{n}\left(\frac{x_i^2\lambda^5}{2\lambda^2+24\beta^2+12\beta\lambda}(1+\beta x_i)^2 e^{-\lambda x_i}\right)$$
$$= \prod_{i=1}^{n}\left(\frac{x_i^2\lambda^5}{2\lambda^2+24\beta^2+12\beta\lambda}\left(1+\beta^2x_i^2+2\beta x_i\right)e^{-\lambda x_i}\right)$$
$$L(x) = \prod_{i=1}^{n}\left(\frac{x_i^2\lambda^5}{(2\lambda^2+24\beta^2+12\beta\lambda)}\left(1+\beta^2x_i^2+2\beta x_i\right)e^{-\lambda x_i}\right)$$
$$= \frac{\lambda^{5n}}{(2\lambda^2+24\beta^2+12\beta\lambda)^n}\prod_{i=1}^{n}\left(x_i^2\left(1+\beta^2x_i^2+2\beta x_i\right)e^{-\lambda x_i}\right).$$

The log-likelihood function is given by

$$\log L = 5n\log\lambda - n\log\left(2\lambda^2+24\beta^2+12\beta\lambda\right)$$
$$+ 2\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\left(1+\beta^2x_i^2+2\beta x_i\right) - \lambda\sum_{i=1}^{n}x_i.$$

Now, differentiating the log-likelihood with respect to parameters λ and β, we obtain the normal equations,

$$\frac{\partial\log L}{\partial\lambda} = \frac{5n}{\lambda} - n\left(\frac{4\lambda+12\beta}{2\lambda^2+24\beta^2+12\beta\lambda}\right) - \sum_{i=1}^{n}x_i = 0,$$
$$\frac{\partial\log L}{\partial\beta} = -n\left(\frac{48\beta+12\lambda}{2\lambda^2+24\beta^2+12\beta\lambda}\right) + \sum_{i=1}^{n}\left(\frac{2\beta x_i^2+2x_i}{1+\beta^2x_i^2+2\beta x_i}\right) = 0.$$

Because of the complicated form of the above likelihood equations, algebraically it is very difficult to solve the system of nonlinear equations. Therefore, we use numerical techniques like the Newton-Raphson method for estimating the parameters of the proposed distribution.

In order to use the asymptotic normality results for determining the confidence interval, we have that if $\hat{\lambda} = (\hat{\theta}, \hat{\alpha})$ denotes the MLE of $\lambda = (\theta, \alpha)$, then we can show

$$\sqrt{n}(\hat{\lambda}-\lambda) \xrightarrow{d} \mathcal{N}_2\left(0, I^{-1}(\lambda)\right),$$

where $I^{-1}(\lambda)$ is the Fisher's information matrix, i.e.,

$$I(\lambda) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2\log L}{\partial\theta^2}\right) & E\left(\frac{\partial^2\log L}{\partial\theta\partial\alpha}\right) \\ E\left(\frac{\partial^2\log L}{\partial\alpha\partial\theta}\right) & E\left(\frac{\partial^2\log L}{\partial\alpha^2}\right)\end{pmatrix},$$

and where

$$E\left(\frac{\partial^2\log L}{\partial\theta^2}\right) = -\frac{5n}{\theta^2} - n\left(\frac{4\left(2\theta^2+24\alpha^2+12\alpha\theta\right)-(4\theta+12\alpha)^2}{\left(2\theta^2+24\alpha^2+12\alpha\theta\right)^2}\right),$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = n\left(\frac{\left(2\theta^2+24\alpha^2+12\alpha\theta\right)(48)-(48\alpha+12\theta)^2}{\left(2\theta^2+24\alpha^2+12\alpha\theta\right)^2}\right) + \sum_{i=1}^{n}\left(\frac{\left(1+\alpha^2x_i^2+2\alpha x_i\right)\left(2x_i^2\right)-\left(2\alpha x_i^2+2x_i\right)^2}{\left(1+\alpha^2x_i^2+2\alpha x_i\right)^2}\right), \quad \text{and}$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) = -n\left(\frac{\left(2\theta^2+24\alpha^2+12\alpha\theta\right)(12)-(4\theta+12\alpha)(48\alpha+12\theta)}{\left(2\theta^2+24\alpha^2+12\alpha\theta\right)^2}\right).$$

Since λ is unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$, and this can be used to obtain asymptotic confidence intervals for θ and α.

9 Application

In this section, we have used a real lifetime data set in the area biased generalized Aradhana distribution to determine its goodness of fit. We then compare this distribution to the generalized Aradhana, quasi Aradhana, Aradhana, and Lindley distributions.

The following real lifetime data set given in Table 1 represents remission time (in months) of 50 breast cancer women subjected to treatment using trastzuzumab as medication reported by the cancer registry department of the University of Benin teaching hospital, Benin, Edo. To estimate the model parameters, the R software is employed. In order to compare the performance of the area biased generalized Aradhana distribution over generalized Aradhana, quasi Aradhana, Aradhana and Lindley distributions, we use the AIC (Akaike Information Criterion),BIC (Bayesian Information Criterion), AICC (Akaike Information CriterionCorrected)and $-2\log L$. The distribution is better for smaller values of AIC,BIC, AICC and $-2\log L$, where

Table 1: Data regarding remission time (in months) of 50 breast cancer women using trastuzumab as medication reported by the University of Benin, Cancer Registry Department

50	74	35	39	21	37	27	35	30	35
26	38	34	34	26	41	61	33	33	26
25	41	35	34	34	33	60	61	42	30
80	31	24	49	26	31	28	41	37	41
61	33	26	34	50	73	45	80	39	21

$$\text{AIC} = 2k - 2\log L,$$
$$\text{BIC} = k\log n - 2\log L,$$
$$\text{AICC} = \text{AIC} + \frac{2k(k+1)}{n-k-1}, \quad \text{and}$$

where n is the sample size, k is the number of parameters in the statistical model, and $-2\log L$ is the maximized value of the log-likelihood function under the considered model.

From Table 2, it is clear that the area biased generalized Aradhana distribution has smaller AIC, BIC, AICC and -2logL values, as compared to generalized Aradhana, quasi Aradhana, Aradhana and Lindley distributions.

Table 2: Performance of Fitted Distributions

Distributions	MLE	S.E	-2logL	AIC	BIC	AICC
Area Biased Generalized Aradhana	$\lambda = 1.262631$ $\beta = 7.324041$	$\lambda = 7.985160$ $\beta = 1.186340$	404.4966	408.4966	412.3206	408.7519
Generalized Aradhana	$\lambda = 7.575677$ $\beta = 3.572997$	$\lambda = 6.184947$ $\beta = 5.305860$	419.3825	423.3825	427.2066	423.6378
Quasi Aradhana	$\lambda = 0.07573918$ $\beta = 0.001000000$	$\lambda = 0.009935162$ $\beta = 0.413967627$	419.4027	423.4027	427.2267	423.6580
Aradhana	$\lambda = 0.073900753$	$\lambda = 0.006033229$	421.2996	423.2996	425.2117	423.3829
Lindley	$\lambda = 0.049318007$	$\lambda = 0.00493249$	437.2288	439.2288	441.1409	439.3121

Hence it can be concluded that the area biased generalized Aradhana distribution leads to a better fit over generalized Aradhana, quasi Aradhana, Aradhana and Lindley distributions.

10 Conclusion

In the present article, a novel distribution termed as area biased generalized Aradhana distribution has been introduced and compared to its baseline and other distributions. Its different statistical properties like moments, shape of the pdf and cdf, harmonic mean, survival function, hazard rate function, reverse hazard rate function, moment generating function, order statistics, Bonferroni and Lorenz curves have been thoroughly studied and explored. Furthermore, its parameters have also been estimated by using the technique of maximum likelihood estimation. Finally, an application of the new distribution has been presented by using a real lifetime data set to demonstrate its significance and supremacy. Hence it is revealed from the result that the proposed area biased generalized Aradhana distribution provides quite satisfactory results over generalized Aradhana, quasi Aradhana, Aradhana and Lindley distributions.

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A Appendix: R Code

```
library(zipfR)
rm(list=ls(all=TRUE))
x=seq(0,15,0.0001)
y=function(x,theta=0.7,alpha=0.6)((x^2)*(theta^5)*(1+alpha*x)^2*exp(-
    theta*x))/(2*(theta^2)+24*(alpha^2)+12*alpha*theta)
plot(x,y(x),"l",col=1,lwd=2,ylab="f(x)",lty=1,ylim=c(0,0.4),sub="Fig.1:
    Pdf plot of Area Biased Generalized Aradhana Distribution")
y=function(x,theta=1.0,alpha=0.8)((x^2)*(theta^5)*(1+alpha*x)^2*exp(-
    theta*x))/(2*(theta^2)+24*(alpha^2)+12*alpha*theta)
curve(y,add=T,col=2,lwd=2,lty=1)
y=function(x,theta=1.3,alpha=1.0)((x^2)*(theta^5)*(1+alpha*x)^2*exp(-
    theta*x))/(2*(theta^2)+24*(alpha^2)+12*alpha*theta)
curve(y,add=T,col=3,lwd=2,lty=1)
y=function(x,theta=1.6,alpha=1.2)((x^2)*(theta^5)*(1+alpha*x)^2*exp(-
    theta*x))/(2*(theta^2)+24*(alpha^2)+12*alpha*theta)
curve(y,add=T,col=4,lwd=2,lty=1)
y=function(x,theta=1.9,alpha=1.4) ((x^2)*(theta^5)*(1+alpha*x)^2*exp(-
    theta*x))/(2*(theta^2)+24*(alpha^2)+12*alpha*theta)
curve(y,add=T,col=6,lwd=2,lty=1)
legend("topright",legend=c(
expression(paste(theta==0.7,",",alpha==0.6)),
expression(paste(theta==1.0,",",alpha==0.8)),
expression(paste(theta==1.3,",",alpha==1.0)),
expression(paste(theta==1.6,",",alpha==1.2)),
expression(paste(theta==1.9,",",alpha==1.4))),lwd=2,col=c(1,2,3,4,6),text
    .width=3.6,cex=0.9,fill=c(1,2,3,4,6))

library(zipfR)
rm(list=ls(all=TRUE))
x=seq(0,10,0.01)
y=function(x,theta=1.6,alpha=0.9) ((theta^2)*Igamma(3,theta*x,lower=TRUE)
    +(alpha^2)*Igamma(5,theta*x,lower=TRUE)+(2*alpha*theta)*Igamma(4,
    theta*x,lower=TRUE))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))
plot(x,y(x),"l",col=2,lwd=2,ylab="F(x)",lty=1,ylim=c(0,1),sub="Fig.2:Cdf
    plot of Area Biased Generalized Aradhana Distribution")
y=function(x,theta=1.9,alpha=1.2) ((theta^2)*Igamma(3,theta*x,lower=TRUE)
    +(alpha^2)*Igamma(5,theta*x,lower=TRUE)+(2*alpha*theta)*Igamma(4,
    theta*x,lower=TRUE))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))
curve(y,add=T,col=3,lwd=2,lty=1)
y=function(x,theta=2.2,alpha=1.5) ((theta^2)*Igamma(3,theta*x,lower=TRUE)
    +(alpha^2)*Igamma(5,theta*x,lower=TRUE)+(2*alpha*theta)*Igamma(4,
    theta*x,lower=TRUE))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))
curve(y,add=T,col=4,lwd=2,lty=1)
y=function(x,theta=2.5,alpha=1.8) ((theta^2)*Igamma(3,theta*x,lower=TRUE)
    +(alpha^2)*Igamma(5,theta*x,lower=TRUE)+(2*alpha*theta)*Igamma(4,
    theta*x,lower=TRUE))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))
curve(y,add=T,col=5,lwd=2,lty=1)
```

```
legend("bottomright",legend=c(
expression(paste(theta==1.6,",",alpha==0.9)),
expression(paste(theta==1.9,",",alpha==1.2)),
expression(paste(theta==2.2,",",alpha==1.5)),
expression(paste(theta==2.5,",",alpha==1.8))),lwd=2,col=c(1,2,3,4,6),text.width=2.4,cex=0.9,fill=c(1,2,3,4,6))

library(zipfR)
rm(list=ls(all=TRUE))
x=seq(0,10,0.01)
y=function(x,theta=1.6,alpha=0.9) 1-((theta^2)*Igamma(3,theta*x,lower=TRUE)+(alpha^2)*Igamma(5,theta*x,lower=TRUE)+(2*alpha*theta)*Igamma(4,theta*x,lower=TRUE))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))
plot(x,y(x),"l",col=2,lwd=2,ylab="R(x)",lty=1,ylim=c(0,1),sub="Fig.3: Reliability plot of Area Biased Generalized Aradhana Distribution")
y=function(x,theta=1.9,alpha=1.2) 1-((theta^2)*Igamma(3,theta*x,lower=TRUE)+(alpha^2)*Igamma(5,theta*x,lower=TRUE)+(2*alpha*theta)*Igamma(4,theta*x,lower=TRUE))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))
curve(y,add=T,col=3,lwd=2,lty=1)
y=function(x,theta=2.2,alpha=1.5) 1-((theta^2)*Igamma(3,theta*x,lower=TRUE)+(alpha^2)*Igamma(5,theta*x,lower=TRUE)+(2*alpha*theta)*Igamma(4,theta*x,lower=TRUE))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))
curve(y,add=T,col=4,lwd=2,lty=1)
y=function(x,theta=2.5,alpha=1.8) 1-((theta^2)*Igamma(3,theta*x,lower=TRUE)+(alpha^2)*Igamma(5,theta*x,lower=TRUE)+(2*alpha*theta)*Igamma(4,theta*x,lower=TRUE))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))
curve(y,add=T,col=5,lwd=2,lty=1)
legend("topright",legend=c(
expression(paste(theta==1.6,",",alpha==0.9)),
expression(paste(theta==1.9,",",alpha==1.2)),
expression(paste(theta==2.2,",",alpha==1.5)),
expression(paste(theta==2.5,",",alpha==1.8))),lwd=2,col=c(1,2,3,4,5),text.width=2.4,cex=0.9,fill=c(1,2,3,4,5))

library(zipfR)
rm(list=ls(all=TRUE))
x=seq(0,0.4,0.0001)
y=function(x,theta=1.0,alpha=0.1) ((x^2)*(theta^5)*(1+alpha*x)^2*exp(-theta*x))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))-((theta^2)*Igamma(3,theta*x,lower=TRUE)+(alpha^2)*Igamma(5,theta*x,lower=TRUE)+2*(alpha)*(theta)*Igamma(4,theta*x,lower=TRUE))
plot(x,y(x),"l",col=1,lwd=2,ylab="h(x)",lty=1,ylim=c(0,0.03),sub="Fig.4: Hazard plot of Area Biased Generalized Aradhana Distribution")
y=function(x,theta=1.2,alpha=0.1) ((x^2)*(theta^5)*(1+alpha*x)^2*exp(-theta*x))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))-((theta^2)*Igamma(3,theta*x,lower=TRUE)+(alpha^2)*Igamma(5,theta*x,lower=TRUE)+2*(alpha)*(theta)*Igamma(4,theta*x,lower=TRUE))
curve(y,add=T,col=2,lwd=2,lty=1)
```

```
y=function(x,theta=1.4,alpha=0.1) ((x^2)*(theta^5)*(1+alpha*x)^2*exp(-
    theta*x))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))-((theta^2)*
    Igamma(3,theta*x,lower=TRUE)+(alpha^2)*Igamma(5,theta*x,lower=TRUE)
    +2*(alpha)*(theta)*Igamma(4,theta*x,lower=TRUE))
curve(y,add=T,col=3,lwd=2,lty=1)
y=function(x,theta=1.6,alpha=0.1) ((x^2)*(theta^5)*(1+alpha*x)^2*exp(-
    theta*x))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))-((theta^2)*
    Igamma(3,theta*x,lower=TRUE)+(alpha^2)*Igamma(5,theta*x,lower=TRUE)
    +2*(alpha)*(theta)*Igamma(4,theta*x,lower=TRUE))
curve(y,add=T,col=4,lwd=2,lty=1)
y=function(x,theta=1.8,alpha=0.1) ((x^2)*(theta^5)*(1+alpha*x)^2*exp(-
    theta*x))/(2*(theta^2)+24*(alpha^2)+12*(alpha)*(theta))-((theta^2)*
    Igamma(3,theta*x,lower=TRUE)+(alpha^2)*Igamma(5,theta*x,lower=TRUE)
    +2*(alpha)*(theta)*Igamma(4,theta*x,lower=TRUE))
curve(y,add=T,col=6,lwd=2,lty=1)
legend("topleft",legend=c(
expression(paste(theta==1.0,",",alpha==0.1)),
expression(paste(theta==1.2,",",alpha==0.1)),
expression(paste(theta==1.4,",",alpha==0.1)),
expression(paste(theta==1.6,",",alpha==0.1)),
expression(paste(theta==1.8,",",alpha==0.1))),lwd=2,col=c(1,2,3,4,6),text
    .width=0.1,cex=0.9,fill=c(1,2,3,4,6))

strength_data=c(50, 74, 35, 39, 21, 37, 27, 35, 30, 35, 26, 38, 34, 34,
    26, 41, 61, 33, 33, 26, 25, 41, 35, 34, 34, 33, 60, 61, 42, 30, 80,
    31, 24, 49, 26, 31, 28, 41, 37, 41, 61, 33, 26, 34, 50, 73, 45, 80,
    39, 21)
dburr=function(x,theta,alpha) ((x^2)*(theta^5)*(1+alpha*x)^2*exp(-theta*x
    ))/(2*(theta^2)+24*(alpha^2)+12*alpha*theta)
library(MASS)
MLE_theta_alpha=fitdistr(x = strength_data,densfun = dburr,start = list(
    alpha=0.5,theta=0.91),lower=c(0.001,0.001),upper=c(Inf,Inf))
MLE_theta_alpha
Minus_2_logL1=-2*fitdistr(x = strength_data,densfun = dburr,start = list(
    alpha=0.5,theta=0.91),lower=c(0.001,0.001),upper=c(Inf,Inf))$loglik
Minus_2_logL1

AIC(MLE_theta_alpha)
BIC(MLE_theta_alpha)
dburr=function(x,alpha=5.478,theta=7.489) ((x^2)*(theta^5)*(1+alpha*x)^2*
    exp(-theta*x))/(2*(theta^2)+24*(alpha^2)+12*alpha*theta)
curve(dburr(x),add=TRUE,col=3,lwd=2)
legend("topright",legend=c("Area Biased Generalized Aradhana Distribution
    ","Generalized Aradhana Distribution","Quasi Aradhana Distribution","
    AradhanaDistribution","Lindley Distribution"),pt.cex=13,cex=0.9,lty
    =2:3,fil=2:3,lwd=2)
dburr=function(x,theta,alpha) ((theta^3)*(1+alpha*x)^2*exp(-theta*x))/((
    theta^2)+2*alpha*theta+2*(alpha^2))
```

```
library(MASS)
MLE_theta_alpha=fitdistr(x = strength_data,densfun = dburr,start = list(alpha=0.5,theta=0.91),lower=c(0.001,0.001),upper=c(Inf,Inf))
MLE_theta_alpha
Minus_2_logL=-2*fitdistr(x = strength_data,densfun = dburr,start = list(alpha=0.5,theta=0.91),lower=c(0.001,0.001),upper=c(Inf,Inf))$loglik
Minus_2_logL

AIC(MLE_theta_alpha)
BIC(MLE_theta_alpha)

hist(strength_data,prob=TRUE,density=c(15),main="Fig.5:Fitting of ABGAD, GAD, Quasi Aradhana, Aradhana and Lindley distribution to the relief time data given in table.1",xlab= "observed")

dburr=function(x,alpha=5.478,theta=7.489)  ((theta^3)*(1+alpha*x)^2*exp(-theta*x))/((theta^2)+2*alpha*theta+2*(alpha^2))
curve(dburr(x),add=TRUE,col=3,lwd=2)

dburr=function(x,theta,alpha) (theta*(alpha+theta*x)^2*exp(-theta*x))/((alpha^2)+2*alpha+2)

library(MASS)
MLE_theta_alpha=fitdistr(x = strength_data,densfun = dburr,start = list(alpha=0.5,theta=0.91),lower=c(0.001,0.001),upper=c(Inf,Inf))
MLE_theta_alpha
Minus_2_logL=-2*fitdistr(x = strength_data,densfun = dburr,start = list(alpha=0.5,theta=0.91),lower=c(0.001,0.001),upper=c(Inf,Inf))$loglik
Minus_2_logL
AIC(MLE_theta_alpha)
BIC(MLE_theta_alpha)
hist(strength_data,prob=TRUE,density=c(15),main="Fig.5:Fitting of  ABGAD, GAD, Quasi Aradhana, Aradhana and Lindley distribution to the relief time data given in table.1",xlab= "observed")
dburr=function(x,alpha=5.478,theta=7.489) (theta*(alpha+theta*x)^2*exp(-theta*x))/((alpha^2)+2*alpha+2)
curve(dburr(x),add=TRUE,col=3,lwd=2)

dburr=function(x,theta) (theta^3*(1+x)^2*exp(-theta*x))/(theta^2+2*theta+2)
```

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