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ON SOME INFERENCES OF RANDOM WALKS

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SUMMARY

In mathematics a random walk (also known as drunkard's walk) is a succession of random steps. In 1905 Karl Pearson introduced the term "random walk". A Bernoulli random walk is the random walk on the integer number line $\mathbb{Z}$ which starts at 0 and at each step moves $+1$ or -1 with equal probability. A Pearsonian random walk is a walk in the plane that starts at the origin 0 and consists of length 1 taken in uniformly random direction. In this paper several known and new results of Bernoulli and Pearsonian walks will be presented.

Keywords and phrases: Random Walk, Generating Function,Central Limit Theorem, Arc Sine Distribution.

Note: This paper is dedicated to the memory of A. K. Md. Ehsanes Saleh.

1 One Dimensional Bernoulli Random Walk

1.1 Introduction

The one dimensional Bernoulli random walk can be considered as follows. At each step one random walker moves one step to the right with probability p, $0 < p < 1$, or one step to the left with probability $1 - p$. Consider the Bernoulli sequences of independent random variables $X_1, X_2, \ldots,$ taking the value 1 with probability p $(0 < p < 1)$ and the value -1 with probability $q = 1 - p$. The events $\{X_n = 1\}$ and $\{X_n = -1\}$ are treated, respectively, as a success and a failure of the t-th Bernoulli trial. Let $S_n = \sum_{i=1}^{n} X_i$. The walker can return to the starting point 0 in even number of steps. If $p > 1/2$, then $S_n \to \infty$ as $n \to \infty$, $S_n \to -\infty$ as $n \to \infty$ if $p < 1/2$ and S_n oscillates between $-\infty$ and $+\infty$ if $p = 1/2$.

2 Main Results

2.1 Asymmetric random walks $(p \neq q)$

Example 2.1.

$$P(S_7 = S_{11}) = P(X_8 + X_9 + X_{10} + X_{11} = 0) = P(S_4 = 0) = \frac{4!}{2!2!} p^2 q^2 = 6p^2 q^2$$

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$$P(S_7=-1,S_{13}=3)=P(X_1+X_2+\cdots+X_7=-1,X_8+X_9+\cdots+X_{13}=4)$$
$$=P(S_7=-1)P(S_6=4)=\frac{7!}{4!3!}p^3q^4\frac{6!}{5!}p^5q=210p^8q^5$$

$$P(S_4=-2,S_{10}=2)=P(X_1+X_2+X_3+X_4=-2,X_5+X_6+...+X_{10}=4)$$
$$=P(S_4=-2)P(S_6=4)=\frac{4!}{3!}pq^3\frac{6!}{5!}p^5q=24p^6q^4$$

We have

$$E(X_i)=2p-1 \text{ and } \mathrm{Var}(X_i)=4p(1-p),\ and$$
$$E(S_n)=n(2p-1),\mathrm{Var}(S_n)=4np(1-p).$$

To return to the starting place 0, there must be even number of steps. Let l_{2n} be the number of steps of length $2n$. We will have

$$l_{2n}=\binom{2n}{n},\quad l_0=1,\ \text{and } nl_{2n}=n\frac{(2n)!}{n!n!}=2(2n-1)\frac{(2n-2)!}{(n-1)!(n-1)!}=2(2n-1)l_{2n-2}.$$

Let $L(s)$ be the generating function of l_{2n}, then

$$L(s)=\sum_{n=0}^{\infty}l_{2n}s^{2n}=\frac{1}{\sqrt{1-4s^2}}.$$

Let $P(s)$ be the probability generating function of the return to the origin 0, then

$$P(s)=\sum_{n=0}^{\infty}l_{2n}p^nq^n=\frac{1}{\sqrt{1-4pq}}.$$

Let t_{2n} be the number of steps to return to 0 for the first time, then

$$l_{2n}=\sum_{k=0}^{n}t_{2k}l_{2n-2k}.$$

Let $T(s)$ be the generating function of t_n, then we will have

$$L(s)-1=L(s)T(s),\ T(s)=1-\frac{1}{L(s)}=1-\sqrt{1-4s^2},\ and\ t_{2n}=\frac{1}{2n-1}\binom{2n}{n}$$

and the probability of returning first time from $2n$ steps is $t_{2n}p^nq^n$. Let $T^*(s)$ be the probability generating function of first time return to the origin, then

$$T^*(s)=1-\sqrt{1-4pqs^2}.$$

The probability of first time return to 0 is

$$T^*(1)=1-\sqrt{1-4pq}=1-|p-q|.$$

Thus if $p = q$, then probability 1 that the walker will return to the origin. The probability p^0 of no return to the origin is $|p-q|$. Let p_k be the probability that the walker will ever reach $x = k$ $(k > 0)$, then by Markov property $p^k = (p_1)^k$. Thus

$$p_1 = p + qp_2 \; and \; p_1 = p + q(p_1)^2.$$

We can write

$$(p_1)^2 - \frac{p_1}{q} + \frac{p}{q} = 0.$$

The solution of the above equation is

$$p_1 = \frac{1}{2q} \pm \frac{1}{2}\sqrt{\frac{1}{q^2} - 4\frac{p}{q}}.$$

Since p_1 is positive, we must have

$$p_1 = \frac{1}{2q} - \frac{1}{2}\sqrt{\frac{1}{q^2} - 4\frac{p}{q}} \; and \; p_1 = \frac{1-\sqrt{1-4pq}}{2q}, \quad p_1 = \begin{cases} 1 & \text{if } p \geq q, \\ p/q & \text{if } p < q. \end{cases}$$

We will have, if $p < q$, then $p^k = \left(\frac{p}{q}\right)^k$ and $p^k = 1$ if $p \geq q$. Let $\phi_X(t)$ be the characteristic function of X_i; then

$$\phi_X(t) = \cos t + i(p-q)\sin t.$$

Let $\phi_{S_n}(t)$ be the characteristic function of $S_n(t)$, then

$$\phi_{S_n}(t) = (\cos t + i(p-q)\sin t)^n.$$

From the above expression, we obtain

$$\mu_1 = E(S_n) = \frac{1}{i}\frac{d}{dt}(\phi_{S_n}(t))\Big|_{t=0} = \frac{1}{i}n(\cos t + i(p-q)\sin t)^{n-1}(-\sin t + i(p-q)\cos t)\Big|_{t=0} = n(p-q)$$

$$\begin{aligned} \mu_2 = E(S_n^2) &= \frac{1}{i^2}\frac{d^2}{dt^2}(\phi_{S_n}(t))\Big|_{t=0} \\ &= \frac{1}{i^2}\Big[n(n-1)(\cos t + i(p-q)\sin t)^{n-2}(-\sin t - i(p-q)\cos t)^2 \\ &\quad + n(\cos t + i(p-q)\sin t)^{n-1}(-\cos t - i(p-q)\sin t)\Big]\Big|_{t=0} = n(n-1)(p-q)^2 + n \\ \text{Var}(S_n) &= n(n-1)(p-q)^2 + n - (n(p-q))^2 = n - n(p-q)^2 = 4npq \end{aligned}$$

$P(S_n = k | S_0 = a) = P(S_n = k + b \,|\, S_0 = n + k).S_{n+m} - S_m$ has the same distribution as $S_n - S_0$.

Lemma 2.1. *If $p \neq 1/2$, then the random walk is transient.*

Proof. Let J_n be the indicator variable that the walker returns to the starting point 0. The total number of visits to the origin is given by $V = \sum_{j=1}^{\infty} J_j$. Now

$$E(V) = \sum_{j=1}^{\infty} E(J_j) = \sum_{n=0}^{\infty} P(S_{2n} = 0) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!n!}p^n q^n = (1-4pq)^{-1/2}. \tag{2.1}$$

For $p \neq 1/2$, $(1-4pq)^{-1/2}$ is finite. Thus the random walk for $p \neq 1/2$ is transient.

2.1.1 The Gambler's ruin problem.

Suppose q_x be the probability of a gambler's x stack reaches zero before it reaches M, given that initial stack is k. We have the condition $q_0 = 1$ and $q_M = 0$. Now

$$q_k = pq_{k+1} + qq_{k-1} \; and \; p(q_{k+1} - q_k) = q(q_k - q_{k-1}), \tag{2.2}$$

i.e., $q_{k+1} - q_k = \frac{q}{p}(q_k - q_{k-1})$. It is easy to see $q_{k+1} - q_k = (\frac{q}{p})^k(q_1 - q_0)$. Now

$$-1 = q_M - q_0 = \sum_{k=0}^{M-1}(q_{k+1} - q_k) = \sum_{k=0}^{M-1}(\frac{q}{p})^k(q_1 - q_0) = (q_1 - q_0)\sum_{k=0}^{M-1}(\frac{q}{p})^k = (q_1 - q_0)\frac{(\frac{q}{p})^M - 1}{(\frac{q}{p}) - 1}.$$

Thus

$$(q_1 - q_0) = -\frac{(\frac{q}{p}) - 1}{(\frac{q}{p})^M - 1}.$$

Now for any $z, 1 \le z \le M$,

$$q_z - q_0 = \sum_{k=0}^{z-1}(q_{k+1} - q_0) = (q_1 - q_0)\sum_{k=0}^{z-1}(\frac{q}{p})^k = (q_1 - q_0)\frac{(\frac{q}{p})^z - 1}{(\frac{q}{p}) - 1} = -\frac{(\frac{q}{p}) - 1}{(\frac{q}{p})^M - 1}\frac{(\frac{q}{p})^z - 1}{(\frac{q}{p}) - 1} = -\frac{(\frac{q}{p})^z - 1}{(\frac{q}{p})^M - 1}.$$

We have

$$q_z = 1 - \frac{(\frac{q}{p})^z - 1}{(\frac{q}{p})^M - 1} = \frac{(\frac{q}{p})^M - (\frac{q}{p})^z}{(\frac{q}{p})^M - 1},$$

for $p = q$, $q_z = (M - z)/M$.

Theorem 1. $\lim_{n\to\infty} \frac{S_n - n(p-q)}{2\sqrt{(npq)}} \to N(0, 1)$.

Proof.

$$\begin{aligned}
E\Big(\exp\Big(i\frac{S_n - n(p-q)}{2\sqrt{(npq)}}\Big)\Big) &= E\Big(\exp\Big(it\frac{\sum_{k=1}^{n}(X_k - p + q)}{2\sqrt{(npq)}}\Big)\Big) = E\Big(\exp\Big(it\sum_{k=1}^{n}\frac{(X_k - p + q)}{2\sqrt{(npq)}}\Big)\Big) \\
&= \Big[E\Big(exp\Big(it\frac{X_k - p + q)}{2\sqrt{(npq)}}\Big)\Big)\Big]^n \\
&= \Big[p\Big\{cos\frac{2q}{2\sqrt{(npq)}}t + i\sin\frac{2q}{2\sqrt{(npq)}}t\Big\} + q\Big\{cos\frac{-2p}{2\sqrt{(npq)}}t - i\sin\frac{-2p}{2\sqrt{(npq)}}t\Big\}\Big]^n \\
&= \Big[p\Big(1 - \frac{(2p)^2}{8npq}t^2\Big) + q\Big(1 - \frac{(2q)^2}{8npq}\Big) + pit\frac{q}{\sqrt{(npq)}} - qit\frac{-2p}{2\sqrt{(npq)}} + e_n\Big]^n,
\end{aligned}$$

where $e_n = 0(n^2) \cdot = \big(1 - \frac{t^2}{2n} + e_n\big)^n$. Now

$$\lim_{n\to\infty}(1 - \frac{t^2}{2n} + e_n)^n = e^{-\frac{t^2}{2}}.$$

Thus $\lim_{n\to\infty} \frac{S_n - n(p-q)}{2\sqrt{(npq)}} \to N(0, 1)$ in distribution.

2.2 Symmetric random variable $p = q$

Table 1 gives an indication of probabilities of one dimensional random walk.

Table 1: Probabilities of 4 symmetric random walk on integer line.

j	-4	-3	-2	-1	0	1	2	3	4
P(S$_0 = j$)					1				
P(S$_1 = j$)				$\frac{1}{2}$		$\frac{1}{2}$			
P(S$_2 = j$)			$\frac{1}{2^2}$		$\frac{2}{2^2}$		$\frac{1}{2^2}$		
P(S$_3 = j$)		$\frac{1}{2^3}$		$\frac{3}{2^3}$		$\frac{3}{2^3}$		$\frac{1}{2^3}$	
P(S$_4 = j$)	$\frac{1}{2^4}$		$\frac{4}{2^4}$		$\frac{6}{2^4}$		$\frac{4}{2^4}$		$\frac{1}{2^4}$

Lemma 2.2 (Weak law of large numbers). $\frac{S_n}{n} \twoheadrightarrow 0$ *as* $n \to \infty$

Proof. Using the Chebyshev's inequality, we have for a given ϵ,

$$P(\left| \frac{S_n}{n} \right| \geq \epsilon) \leq \frac{1}{\epsilon^2} E(\left|\frac{S_n}{n}\right|)^2 = \frac{1}{n\epsilon^2} \twoheadrightarrow 0, \quad as \;\; n \twoheadrightarrow \infty.$$

Theorem 2. $\frac{S_n}{\sqrt{n}} \twoheadrightarrow N(0,1)$ *in distribution as* $n \twoheadrightarrow \infty$.

Proof. Let $\varphi_n(t)$ be the characteristic function of $\frac{S_n}{\sqrt{n}}$. We will have

$$\varphi_n(t) = E\big(e^{\frac{S_n}{\sqrt{n}} it}\big) = E\big(e^{\frac{it}{\sqrt{n}} \sum_{j=1}^{n} X_j}\big) = \big[E\big(e^{\frac{it}{\sqrt{n}} X_j}\big)\big]^n = \big[\cos(\frac{t}{\sqrt{n}})\big]^n$$
$$= \Big[1 - \frac{t^2}{2n} + \frac{t^4}{4!n^2} + \cdots\Big]^n \twoheadrightarrow e^{-\frac{t^2}{2}} \text{ as } n \twoheadrightarrow \infty$$

Thus $\frac{S_n}{\sqrt{n}} \twoheadrightarrow N(0,1)$ in distribution as $n \twoheadrightarrow \infty$.

Theorem 3. $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0) = P(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n} \geq 0)$.

Proof. $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0) = 2P(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0)$

Now

$$\begin{aligned} P(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0) &= P(S_1 = 1)P(S_2 > 0, \ldots, S_{2n} > 0 | S_1 = 1) \\ &= P(S_1 = 1)P(S_2 - S_1 \geq 0, \ldots, S_{2n} - S_1 \geq 0) \\ &= \frac{1}{2} P(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n-1} \geq 0) \end{aligned}$$

Since S_{2n-1} is odd, we must have $S_{2n=1} \geq 1$ and $S_{2n} \geq 0$.

Thus

$$P(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n-1} \geq 0) = \frac{1}{2} P(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n-1} \geq 0, S_{2n} \geq 0.$$

Hence

$$P(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0) = \frac{1}{2} P(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n-1} \geq 0, S_{2n} \geq 0), and$$
$$P(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0) = P(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n} \geq 0).$$

Theorem 4. $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0) = P(S_{2n} = 0)$.

Proof.

$$\begin{aligned} P(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0) &= 2P(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0) \\ &= 2 \sum_{r=1}^{n} P(S_1 > 0, S_2 > 0, \ldots, S_{2n} = 2r) \\ &= 2 \sum_{r=1}^{n} \left\{ \binom{2n-1}{n+r-1} - \binom{2n-1}{n+r} \right\} \frac{1}{2^{2n}} \\ &= 2 \binom{2n-1}{n} \frac{1}{2^{2n}} = \binom{2n}{n} \frac{1}{2^{2n}} = P(S_{2n} \neq 0). \end{aligned}$$

Let $N_n(a, b)$ be the number of path from a, b in n steps. $N_n^+(a, b)$ be the number of path from a, b in n steps without visiting 0.

Theorem 5. $N_n^+(0, k) = \frac{k}{n} N_n(0, k)$.

Proof.

$$N_n^+(0, k) = N_{n-1}(1, k) - N_{n-1}(-1, k) = \binom{n-1}{\frac{(n+k)}{2} - 1} - \binom{n-1}{\frac{(n+k)}{2}}$$

Let $((n + k)/2 = v$, then

$$\begin{aligned} N_n^+(0, k) &= \binom{n-1}{v-1} - \binom{n-1}{v} = \frac{(n-1)!}{(v-1)!(n-v)!} - \frac{(n-1)!}{v!(n-v-1)!} \\ &= \frac{v}{n} \frac{n!}{v!(n-v)!} - \frac{n-v}{n} \frac{n!}{v!(n-v)!} = \frac{n!}{v!(n-v)!} \left(\frac{v}{n} - \frac{n-v}{n}\right) = \frac{n!}{v!(n-v)!} \frac{2v-n}{n} \\ &= \frac{k}{n} \frac{n!}{v!(n-v)!} = \frac{k}{n} \frac{n!}{(\frac{n+k}{2})!(\frac{n-k}{2})!} = \frac{k}{n} N_n(0, k) \end{aligned}$$

Theorem 6. *Let E_n be the expected distance of walker in n steps, then*

$$|E_{2n}| = \frac{(2n-1)!!}{(2n-2)!!} \ \ and \ \ |E_{2n+1}| = \frac{(2n+1)!!}{(2n)!!},$$

where $x!! = 1(3)(5) \cdots x$, if x is odd and $x!! = 2(4)(6) \cdots x$, if x is even.

Proof.

$$|E_{2n}| = \sum_{k=1}^{n} 2kP(|S_{2n}| = 2k) = \sum_{k=1}^{n} \frac{(2n)!}{(n+k)!(n-k)!} \frac{k}{2^{2n-2}}$$
$$= \frac{(2n)!}{2^{2n-1}} \sum_{k=1}^{n} \frac{2k}{(n+k)!(n-k)!} = \frac{(2n)!}{2^{2n-1}} \frac{n+1}{(n+1)!(n-1)} = \frac{(2n-1)!!}{(2n-2)!!}.$$

and

$$|E_{2n+1}| = \sum_{k=1}^{n} (2k+1)P(|S_{2n+1}| = 2k+1) = \sum_{k=0}^{n} \frac{2(2n+1)!}{(n+k+1)!(n-k)!} \frac{2k+1}{2^{2n+1}}$$
$$= \frac{(2n+1)!}{2^{2n}} \sum_{k=0}^{n} \frac{2k+1}{(n+k+1)!(n-k)!} = \frac{(2n+1)!}{2^{2n}} \frac{1}{(n!)^2} = \frac{(2n+!)!!}{(2n)!!}$$

$|E_{2n}|$ and $|E_{2n+1}|$ are the coefficients respectively in the series $(1-x)^{-3/2}$, where

$$(1-x)^{-3/2} = 1 + \frac{3}{2}x + \frac{15}{8}x^2 + \cdots + \frac{(2n-1)!!}{(2n-2)!!}x^{n=1} + \frac{(2n+!)!!}{(2n)!!}x^n + \cdots$$

Theorem 7. *Let f_{2n} be the probability to return to 0 for the first time in $2n$ steps and u_{2n} be the probability to return to 0 in 2n steps. We have*

$$u_{2n} = f_0 u_{2n} + f_2 u_{2n-2} + \cdots + f_{2n} u_0$$

Let $F(s)$ and $U(s)$ be the generating functions of f_{2n} and u_{2n} respectively. Then $U(s) = U(s)F(s) + 1$ and $F(s) = 1 - \frac{1}{U(s)}$,

$$U(s) = \sum_{0}^{\infty} \frac{(2n)!}{n!n!} \frac{s^n}{2^{2n}} = (1-s)^{1/2},$$

u_{2n} is the coefficient of s^n from U(s) and it is $\frac{(2n)!}{n!n!}\frac{1}{2^{2n}}$,

$$F(s) = 1 - (1-s)^{1/2},$$

where f_{2n} is the coefficient of s^n from F(s) and it is $\frac{1}{2n-1}\frac{(2n)!}{n!n!} \cdot \frac{1}{2^{2n}}$. For large n,

$$f_{2n} \backsim \frac{1}{2n-1} \frac{1}{\sqrt{\pi n}}.$$

Let τ_0 be the first time that the walker returns to origin 0 in $2n$ steps. Then $P(\tau_0 = 2n) = f_{2n} = \frac{1}{2n-1}\frac{1}{\sqrt{\pi n}}$. Thus,

$$P(P_{\tau_0} \geq r) = \int_r^{\infty} \frac{1}{2}\sqrt{\frac{1}{\pi}} x^{-3/2} dx = \frac{1}{\sqrt{(\pi r)}} \leq \frac{1}{\sqrt{r}}.$$

Theorem 8.

$$f_{2n} = u_{2n-2} - u_{2n}$$

Proof. We have,

$$\begin{aligned} u_{2n-2} - u_{2n} &= \binom{2n-2}{n-1}\frac{1}{2^{2n-2}} - \binom{2n}{n}\frac{1}{2^{2n}} = \frac{(2n-2)!}{(n-1)!(n-1)}\frac{1}{2^{2n-2}} - \frac{(2n)!}{(n)!(n)}\frac{1}{2^{2n}} \\ &= \frac{(2n-2)!}{(n-1)!(n-1)!}\frac{1}{2^{2n-2}}(1 - \frac{(2n)(2n-1)}{4n^2}) \\ &= \frac{(2n-2)!}{(n-1)!(n-1)!}\frac{1}{2^{2n-2}}\frac{1}{2n} = \frac{(2n)!}{(2n-1)}.\frac{1}{n!n1}\frac{1}{2^2} = \frac{1}{2n-1}u_{2n} = f_{2n}. \end{aligned}$$

We can write $u_{2n} = \binom{2n}{n}\frac{1}{2^{2n}} = (-1)^n\binom{-\frac{1}{2}}{n}$; where

$$\begin{aligned} (-1)^n\binom{-\frac{1}{2}}{n} &= \frac{(-1)^n(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)(-\frac{1}{2}-3)\cdots(-\frac{1}{2}-n+1)}{n!}, \; n = 2,4,6,\ldots \\ f_{2n} &= \frac{1}{2n-1}\frac{(2n)!}{n!n!}\frac{1}{2^{2n}} = (-1)^{n+1}\binom{\frac{1}{2}}{n}, \\ (-1)^{n+1}\binom{-\frac{1}{2}}{n} &= \frac{(1)^{n+1}(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)\cdots(\frac{1}{2}-n+1)}{n!}, n = 2,4,6,\ldots \end{aligned}$$

The following theorem gives an alternative proof of Theorem 3.

Theorem 9. $P(S_1 \neq 0, S_2 \neq 0, \ldots, S_n \neq 0) = P(S_{2n} = 0)$.

Proof.

$$1 - (f_2 + f_4 + \cdots + f_{2n}) = 1 - (1 - u_2 + u_2 - u_4 + \cdots + u_{2n-2} - u_{2n}) = u_{2n}.$$

Thus

$$P(S_1 \neq 0, S_2 \neq 0, \cdots, S_n \neq 0) = P(S_{2n} = 0).$$

Theorem 10. *Let W be the waiting time of first return to the origin 0, then*

$$E(W) = \sum_{n=1}^{\infty} 2nf_{2n} = \sum_{n=1}^{\infty}\frac{2n}{2n-1}\frac{(2n)!}{n!n!} \sim \sum_{n=2}^{\infty}\frac{1}{\sqrt{\pi n}} \geq \sum_{n=2}^{\infty}\frac{1}{n\sqrt{\pi}} = \infty.$$

Theorem 11. *Let A_k be the first visit from 0 to k $(k > 0)$ in n steps, then* $P(A_k) = \frac{k}{n}\binom{n}{(n+k)/2}\frac{1}{2^n}$.

Proof. It follows from Theorem 3 that

$$P(A_k) = \frac{k}{n}P(S_n = k) = \frac{k}{n}\binom{n}{(n+k)/2}\frac{1}{2^n}.$$

Note, $n - k > 0$ and $n - k$ is even. Let

$$\begin{aligned} \alpha_n &= P(S_1 \leq 0, S_2 \leq 0, \ldots, S_{n-1} \leq 0, S_n > 0) \\ \beta_n &= P(S_1 \geq 0, S_2 \geq 0, \ldots, S_{n-1} \geq 0, S_n < 0), \; n = 1,3,5,\ldots \end{aligned}$$

then

$$\alpha_{2n+1} = \beta_{2n+1} = \frac{(2n)!}{n!n!} \frac{1}{2^{2n+1}} \frac{1}{n+1} = \frac{C(n)}{2^{2n+1}},$$

where $C(n)$ is the catalan number. It can be shown easily that $\sum_{n=0}^{\infty} \frac{C(n)}{2^{2n+1}} = 1$ The catalan number $C(n)$ is the number of steps of the simple random walk leading to the first epoch. The generating $A(s)$ functions of α_n and of β_n) is

$$A(s) = \sum_{n=0}^{\infty} \alpha_{2n+1} s^{2n+1} = \frac{1 - \sqrt{(1 - s^2)}}{s}.$$

Theorem 12. *Suppose $X_n, n = 1, 2, \dots$ be a simple symmetric random walk on the one dimension integer lattice $\mathbb{Z}$ and $(F_n)_{n \geq 0}$ be natural filtration, then the followings are true:*

(i) $Y_n = S_n^2 - n$ *is a martingale.*

(ii) $Z_n = S_n^3 - 3nS_n$ *is a martingale.*

Proof. (i)

$$E(X_{n+1}|F_n) = E(X_n + X_{n+1}|F_n) = X_n$$
$$E(Y_{n+1}|F_n) = E(S_{n+1}^2 - (n+1)|F_n) = E((S_n^2 + 2S_nX_{n+1} + X_{n+1}^2 - n - 1)|F_n) = S_n^2 - n = Y^n$$

(ii)

$$\begin{aligned} E(Z_{n+1}|F_n) &= E(S_{n+1}^3 - 3(n+1)S_{n+1}|F_n) = E((S_n + X_{n+1})^3 - 3(n+1)(S_n + X_{n+1})|F_n) \\ &= E(S_n^3 + 3S_n^2X_{n+1} + 3S_nX_{n+1}^2 + X_{n+1}^3 - 3(n+1)(S_n + X_{n+1})|F_n) \\ &= S_n^3 - 3nS_n = Z_n. \end{aligned}$$

2.3 The maximum and minimum of a symmetric random walk

Let

$$M^n = \max(0, S_1, S_2, \dots, S_n) \text{ and } m^n = \min(0, S_1, S_2, \dots, S_n).$$

The following theorem was given by Gut (2008).

Theorem 13. *(i) If the random walk drifts to $+\infty$, then*

$$M^n \xrightarrow{a.s.} +\infty \text{ and } m^n \xrightarrow{a.s.} m = \min S_n > -\infty, \text{ as } n \to \infty,$$

(ii) If the random walk drifts to $-\infty$, then

$$M^n \xrightarrow{a.s.} M = \sup_{n \geq 0} S_n < \infty \text{ a.s. and } m^n \xrightarrow{a.s.} -\infty, \text{ as } n \to \infty,$$

(iii) If the random walk is oscillating, then

$$M^n \xrightarrow{a.s.} +\infty \text{ and } m^n \xrightarrow{a.s.} -\infty, \text{ as } n \to \infty.$$

Pólya (1921) gave the following excellent theorem.

Theorem 14. *A simple (symmetric) random walk in dimension $\mathbb{Z}^d$ is recurrent if $d = 1$ or 2 and transient if $d \geq 3$.*

Proof. Let $d = 1$, then

$$P(S_{2n} = 0) = \frac{(2n)!}{n!n!} \frac{1}{2^{2n}}.$$

Using Stirling's approximation to $n!$, we have

$$\frac{(2n)!}{n!n!} \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}.$$

Therefore

$$\sum_{n=0}^{\infty} P(S_{2n} = 0) = \infty.$$

Thus the random walk is recurrent. Let $d = 2$. Let $S_{2n}^2 = 0$ be the number of steps of the walker to return to the starting point 0, then

$$P(S_{2n}^2 = 0) = \frac{1}{4^{2n}} \sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!} = \frac{1}{4^{2n}} \binom{2n}{n}^2$$

Using Stirling's approximation, we have

$$\frac{1}{4^{2n}} \binom{2n}{n}^2 \sim \frac{1}{\pi n}$$

Thus

$$\sum_{n=0}^{\infty} P(S_{2n}^2 = 0) = \infty.$$

Hence the random walk is recurrent. Let $d = 3$, then

$$P(S_{2n}^3 = 0) = \frac{1}{2^{2n}} (2n)! \sum_{j,k} \left(\frac{1}{3^n} \frac{n!}{j!k!(n-j-k)!} \right)^2 \leq \frac{1}{2^{2n}} (2n)! \frac{n!}{3^n \left(\frac{n}{3}\right)!^3} \leq \frac{c}{n^{3/2}},$$

where c is a constant. Thus

$$\sum_{n=1}^{\infty} P(S_{2n}^3 = 0) < \infty.$$

Hence the random walk is transient. Let $d > 3$. Suppose u_{2n}^d be the probability of the walker to return to the origin 0. It can be shown that $u_{2n}^{2d} < u_{2n}^3$. Thus

$$\sum_{k=1}^{\infty} u_{2n}^{2d} < \sum_{k=1}^{\infty} u_{2n}^3 < \infty.$$

Thus the random walk for $d > 3$ dimensions is transient. □

3 Travel Times

Let $E^{j,k} = E$ (the time taken to go from j to k). We have

$$E_{0,1} = 1$$
$$E_{1,2} = 1 + \frac{1}{2}E_{0,2} = 1 + \frac{1}{2}(E_{0,1} + E_{1,2}) = \frac{3}{2} + \frac{1}{2}E_{1,2}$$
$$E_{1,2} = 3 \text{ and } E_{0,2} = 4.$$

We have

$$E_{2,3} = 1 + \frac{1}{2}E_{1,3} = 1 + \frac{1}{2}(E_{1,2} + E_{2,3}) = \frac{5}{2} + \frac{1}{2}E_{2,3}$$

So $E_{2,3} = 5$. Thus for $k = 1, 2, 3$, we have

$$E_{k-1,k} = 2k - 1, \quad E_{0,k} = k^2$$

Suppose that relation holds for $k \leq n$,

$$E_{n,n+1} = 1 + \frac{1}{2}E_{n-1,n+1} = 1 + \frac{1}{2}(E_{n-1,n} + E_{n,n+1}) = 1 + \frac{2n-1}{2} + \frac{1}{2}E_{n,n+1}.$$

Thus

$$E_{n,n+1} = 2n + 1$$
$$E_{0,n+1} = E_{0,n} + E_{n,n+1} = n^2 + 2n + 1 = (n+1)^2.$$

4 Characteristic Function

Let $S_n^d = X_1 + X_2 + \ldots + X_n$ be for a simple random walk in dimension $\mathbb{Z}^d$. We have the characteristic function $\phi_{X(1)}(\theta)$ of X_1 is

$$\phi_{X(1)}(\theta) = E(e^{iX(\theta)}) = \frac{1}{2d}\sum_{i=1}^{d}(e^{i\theta_i} + e^{-i\theta_i}) = \frac{1}{d}\sum_{i=1}^{d}\cos\theta_i.$$

The characteristic function $\phi_{S(n)}(\theta)$ of S_n^d is

$$E(e^{iS_n^d\theta}) = \left(\frac{1}{d}\sum_{i=1}^{d}\cos\theta_i\right)^n.$$

Let S_n^d be the position of the walker at the n-th step, then

$$P(S_n^d = x) = \left(\frac{1}{2\pi}\right)^n \int e^{-ix\theta}\phi_{S(n)}(\theta)d\theta$$
$$P(S_{2n}^d = 0) = \left(\frac{1}{2\pi}\right)^n \int_{-\pi}^{\pi}\phi_{S(n)}(\theta)d\theta = \left(\frac{1}{2\pi}\right)^n \int_{-\pi}^{\pi}\left(\frac{1}{d}\sum_{i=1}^{d}\cos\theta_i\right)^{2n} d\theta;$$
$$P(S_{2n} = 0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^{2n}\theta d\theta = \frac{(2n)!}{n!n!}\cdot\frac{1}{2^{n+1}}.$$

For $d = 1$, we obtain $\phi_{S(n)}(t)$ as

$$\phi_{S(n)}(t) = \cos^n \theta$$

Differentiating the above expression and putting $\theta = 0$, we obtain

$$\begin{aligned}
&\mu_1 = 0, \mu_2 = n, \mu_3 = 0, \mu_4 = n(3n-2), \mu_5 = 0, \mu_6 = n(15n^2 - 30n + 16),\\
&\mu_7 = 0, \mu_8 = n(105n^3 - 420n^2 + 585n - 272), \mu_9 = 0 \text{ and}\\
&\mu_{10} = n(945n^4 - 6300n^3 + 18960n^2 - 23820n + 7936).
\end{aligned}$$

It is easy to see that $\frac{\mu_r}{n^{r/2}}$ coincides with the moments of $N(0,1)$.

$$\begin{aligned}
P(S^n = j) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\theta} \cos^n \theta d\theta = \frac{1}{2^{n+1}}(1 + (-1)^{n+j}) \frac{n!}{\left(\frac{n+j}{2}\right)! \left(\frac{n-j}{2}\right)!}\\
P(S^{2n} = 0) &= \frac{1}{2^{2n}} \frac{(2n)!}{n!n!}.\\
P(S^{2n} \geq \sqrt{2n}\ln n) &= \frac{\text{Var}\big(|S_{2n}|\big)}{2n\ln^2 n} = \frac{1}{\ln^2 n} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Lemma 4.1. *If $X = (X_1, X_2, \ldots, X_d)$ is a $\mathbb{Z}^d$ valued random vector, then*

$$P(X = x) = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} e^{-ix\theta} \phi_X^d(\theta) d\theta.$$

Proof. Since $\phi_X^d(\theta) = E(e^{ix\theta}) = \sum_{y\in\mathbb{Z}^d} P(X = y)e^{iy\theta}$, we have

$$\int_{(-\pi,\pi)^d} e^{-ix\theta}\phi_X^d(\theta)d\theta = \int_{(-\pi,\pi)^d} e^{-ix\theta} \sum_{y\in\mathbb{Z}^d} P(X = y)e^{iy\theta}d\theta$$

Interchanging the sum and integrals, we obtain

$$\int_{(-\pi,\pi)^d} e^{-ix\theta}\phi_X^d(\theta)d\theta = \sum_{y\in\mathbb{Z}^d} P(X = y) \int_{(-\pi,\pi)^d} e^{-i(y-x)\theta}d\theta$$

and if $x, y \in \mathbb{Z}^d$, then

$$\int_{(-\pi,\pi)^d} e^{-i(y-x)\theta}d\theta = \begin{cases} (2\pi)^d, & \text{if } y = x\\ 0 & \text{if } y \neq x. \end{cases}$$

Thus

$$P(X = x) = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} e^{-ix\theta}\phi_X^d(\theta)d\theta.$$

□

5 Brownian Motion

A stochastic (one dimensional) Wiener process (also called Brownian motion) is a stochastic process (W_t) indexed by non-negative numbers t with the following properties:

(i) $W_0 = 0$.

(ii) With probability 1, W_t is continuous in t.

(iii) The process (W_t) has stationary independent increments.

(iv) The increment $W_{t+s} - W_s$ has the normal $N(0, t)$ distribution.

Consider the random walks independent variables, X_i, $i = 1, 2, \dots$ with as follows

$$X_i = \sqrt{\delta} \text{ with probability } (1/2) \text{ and } = -\sqrt{\delta} \text{ with probability } (1/2)$$
$$E(X_i) = 0 \; and \; \text{Var}(X_i) = \delta.$$

We define the process W_t as follows. Let $W_0 = 0$ and $n\delta = t$, $W_t = W_{n\delta} = \sum_{i=1}^{n} X_i$.

We have

$$E(W_t) = 0 \; and \; \text{Var}(W_t) = n\delta = t.$$

We can now take the continuous limit to see that the random walk W_t converges to a continuous stochastic process called Brownian motion. For any $t \in (0, \infty)$, as $n \to \infty$, $\delta \to 0$ and $n\delta = t$. Thus by central limit theorem the distribution of $W(t)$ is $N(0, t)$.

For $t_1 = n_1\delta$ and $t_2 = n_2\delta$ for $0 \leq t_1 < t_2$, we have

$$W(t_1) = W(n_1\delta) = \sum_{i=1}^{n_1} X_i, \; W(t_2) = W(n_2\delta) = \sum_{i=1}^{n_2} X_i$$
$$W(t_2) - W(t_1) = \sum_{i=n_1+1}^{n_2} X_i.$$

Therefore

$$E(W(t_2) - W(t_1)) = 0 \;\; and \;\; \text{Var}(W(t_2) - W(t_1)) = (n_2 - n_1)\delta = t_2 - t_1.$$

Thus $W(t)$ has stationary independent increments and for any $t \in (0, \infty)$ and $s \in (0, \infty)$, the distribution of $W_{t+s} - W_s$ is $N(0, t)$ distribution. Hence $W(t)$ is a Brownian motion.

6 Pearsonian Random Walk

In 1905 Pearson proposed the following: ”A man starts from a point 0 and walks one step in a straight line, then he turns any angle whatever and walks one step in a straight line. I require the probability that after n steps he will be at a distance r and $r + dr$ from the starting point 0”. Let $X(n)$ be the distance traveled in n steps.

Kluyver (1908) gave the probability density (pdf) of $X(n)$. The pdf $p_n(x)$ of $X(n)$ is given as

$$p_n(x) = \int_0^n xtJ_0(xt)(J_0(t))^n dt, \quad 0 \le x \le n, \tag{6.1}$$

where $J_0(t)$ is the Bessel function of first kind and zero order.

The solution of (6.1) for $n = 2$ is

$$p_2(x) = \frac{2}{\pi}(4 - x^2)^{-1/2}, \quad 0 \le x \le 2.$$

For $n = 3$:

$$p_3(x) = \frac{2\sqrt{3}}{\pi} \cdot \frac{x}{3x^2} \cdot {}_2F_1\Big[\frac{\frac{1}{3}, \frac{2}{3}}{1}(\frac{x^2 + 3x^2}{3x^2}), \quad 0 \le x \le 3,\Big]$$

For $n = 4$:

$$p_4(x) = \frac{2}{x^3} \cdot \frac{\sqrt{16 - x^2}}{x} \cdot {}_3F_2\left[\frac{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}{\frac{5}{6}, \frac{7}{6}}\left(\frac{(16 - x^2)^4}{108x^4}\right)\right], \quad 0 \le x \le 4,$$

where

$${}_pF_q\left(\frac{a_1, a_2, \ldots, a_p}{\beta_1, \beta_2, \ldots, \beta_q}(x)\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(\beta_1)_k (\beta_2)_k \ldots (\beta_q)_k} \frac{x^k}{k!}$$

with $(a)_k = a(a + 1) \ldots (a + k - 1)$.

Rayleigh (1905a, 1905b) showed that for $n \ge 5$, $p_n(x)$ is close to $f_n(x)$, where

$$f_n(x) = \frac{2x}{n} e^{-\frac{x^2}{n}}, \quad 0 \le x \le n.$$

Let μ'_r be the r-th moment of $X(2)$, then

$$\mu'_r = \int_0^2 x^r \frac{2}{\pi}(4 - x^2)^{-1/2} dx = \int_0^1 \frac{2}{\pi} 2^r w^r (1 - w^2)^{-1/2} dw$$
$$= \frac{2^r}{\pi}\Gamma\left(\frac{1}{2}r + \frac{1}{2}\right)\Gamma(1)/\Gamma\left(\frac{1}{2}r + 1\right)$$

For $r = 2m$

$$\mu_2^{2m} = \frac{2^{2m}}{\pi}\Gamma\left(m + \frac{1}{2}\right)/\Gamma(m + 1) = \frac{(2m)!}{m!m!}$$

Table 2: Moments of μ'_2.

r	1	2	3	4	5	6	7	8	9	10
μ'_2	0.31831	2	3.3953	6	10.865	20	37.251	70	132.45	252

We can also write

$$\mu'_2 = \int_0^1 \frac{2}{\pi} 2^r w^r (1 - w^2)^{-1/2} dw$$

Let $u = \cos \pi\theta/2$, then

$$\mu_2' = \int_0^1 2^r \cos^r \pi\theta/2 d\theta = \int_0^1 (4 - 4\sin^2 \pi\theta/2)^{r/2} d\theta$$
$$= \int_0^1 (2 + 2\cos 2\pi\theta)^{r/2} d\theta = \int_0^1 [1 + e^{2\pi i\theta}]^{r/2} d\theta$$

Borwein et al. (2010) showed that for any $n \geq 2$,

$$\mu_n^r = \int_0^1 \int_0^1 \cdots \int_0^1 \left[1 + \sum_{k=1}^{n-1} e^{2\pi i k x_1}\right]^{r/2} dx_1 dx_2 \ldots dx_{n-1}$$

Now

$$\left|1 + \sum_{k=1}^{n-1} e^{2\pi i k x_i}\right| = \Big[(1 + \cos 2\pi x_1 + \cos 2\pi x_2 + \cdots + \cos 2\pi x_{n-1})^2$$
$$+ (\sin 2\pi x_1 + \sin 2\pi x_2 + \cdots + \sin 2\pi x_{n-1})^2\Big]^{1/2}$$
$$= \Big[n + 2 \sum_{1 \leq i < j \leq n-1} \cos 2\pi x_i \cos 2\pi x_j$$
$$+ 2 \sum_{1 \leq i < j \leq n-1} \sin 2\pi x_i \sin 2\pi x_j + 2 \sum_{j=1}^{n-1} \cos 2\pi x_j\Big]^{1/2}$$
$$= \Big[n + 2 \sum_{1 \leq i < j \leq n-1} \cos 2\pi(x_i - x_j) + 2 \sum_{j=1}^{n-1} \cos 2\pi x_j\Big]^{1/2}$$
$$= \Big[n + 2 \sum_{1 \leq i < j \leq n-1} (1 - 2\sin^2 \pi(x_i - x_j)) + 2 \sum_{j=1}^{n-1} (1 - 2\sin^2 \pi x_j)\Big]^{1/2}$$
$$= \Big[n^2 - 4 \sum_{1 \leq i < j \leq n-1} \sin^2 \pi(x_i - x_j) - 4 \sum_{j=1}^{n-1} \sin^2 \pi x_j\Big]^{1/2}.$$

Thus

$$\mu_n^r = \int_0^1 \cdots \int_0^1 \Big[n^2 - 4 \sum_{1 \leq i < j \leq n-1} \sin^2 \pi(x_i - x_j) - 4 \sum_{j=1}^{n-1} \sin^2 \pi x_j\Big]^{r/2} dx_1 \ldots dx_{n-1}. \quad (6.2)$$

For $n = 3$, we have the r-th moment μ_3^r as

$$\mu_3^r = \int_0^1 \int_0^1 \left[1 + e^{2\pi i k x_1} + e^{2\pi i x_2}\right]^{r/2} dx_1 dx_2$$
$$= \int_0^1 \int_0^1 \left[9 - 4(\sin^2 \pi x_1 + \sin^2 \pi x_2 + \sin^2 \pi(x_1 - x_2))\right]^{r/2} dx_1 dx_2.$$

For even moments $m \geq 1$, $0 \leq m_1, m_2, m_3 \leq m$,

$$\mu_3^{2m} = \sum_{m_1+m_2+m_3=m} \left(\frac{m!}{m_1!m_2!m_3!} \right)^2$$

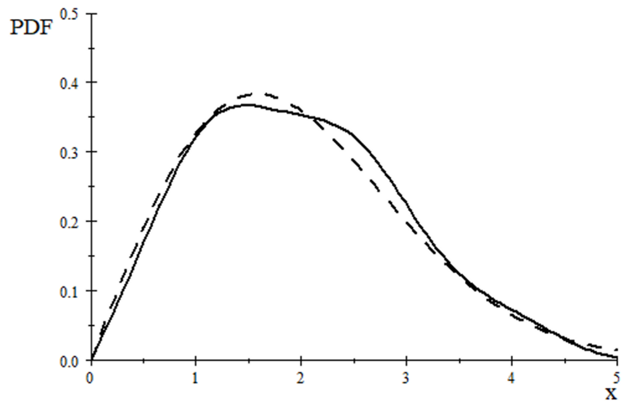

Figure 1: PDF $p_5(x)$ (solid) and $f_5(x)$ (dash)

Table 3: Moments μ_3^r

r	1	2	3	4	5	6	7	8	9	10
μ_3^r	1.5732	3	5.3417	15	36.7052	93	241.5440	639	1714.62	4653

For even moments $m \geq 1$,

$$\mu_n^{2m} = \sum_{m_1+m_2+\cdots+m_n=m} \left(\frac{m!}{m_1!m_2!\cdots m_n!} \right)^2, \; where \; 0 \leq m_1, m_2, \ldots, m_n \leq m.$$

On simplification we obtain

$$\begin{aligned} \mu_n^2 &= n, \\ \mu_n^4 &= 2n^2 - n \\ \mu_n^6 &= 6n^3 - 9n^2 + 4n \\ \mu_n^8 &= 24n^4 - 72n^3 + 96n^2 - 33n \\ \mu_n^{10} &= 120n^5 - 600n^4 + 1250n^3 - 1225n^2 + 456n. \end{aligned}$$

Straub (2010) gave the following alternative for r-th moment of PWR(n):

$$\mu_n^r = \int_0^1 \int_0^1 \cdots \int_0^1 \left[n^2 - 4 \left(\sum_{i<j\leq n} \sin^2 \pi(x_i - x_j) \right) \right]^{r/2} dx_1 dx_2 \ldots dx_n \qquad (6.3)$$

The integrals given in (6.2) and (6.3) will give the same result.

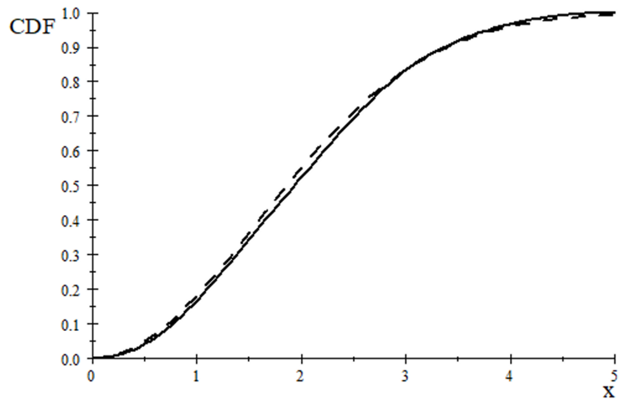

Figure 2: CDFs $P_5(x)$ (solid) and $F_5(x)$ (dash)

For some interesting results on Pearsonian random walk, see Borwein et al. (2010). The graphs in Figures 1 and 2 were produced using Scientific Workplace 5.5.

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