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ADVANCEMENTS IN SHRINKAGE ESTIMATION UTILIZING ROBUST PARAMETERS FOR THE BIRNBAUM-SAUNDERS DISTRIBUTION IN THE CASE OF MULTIPLE SAMPLES

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SUMMARY

In this study, we expanded the improved estimation strategies for robust estimators of the Birnbaum-Saunders distribution for the shape parameter for multiple samples while integrating sample and uncertain prior information. We have used the following estimators: the Graybill-Deal type estimator, the linear shrinkage estimator, the pretest estimator, the shrinkage preliminary estimator, the James-stein and positive James-stein estimation techniques. We developed a test statistic to accept or reject the null hypothesis when considering uncertain prior information. We also explored the asymptotic properties of the proposed estimators. To evaluate their effectiveness, we conducted Monte Carlo simulations using various parameter values and sample sizes that align with our theoretical findings. Additionally, we included a real data example to illustrate the estimators performance in real life application.

Keywords and phrases: Asymptotic Distribution; Fatigue Life Distribution; Robust estimators; Shrinkage estimators; Simulation

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1 Introduction

The Birnbaum-Saunders (BirSau) distribution is a probability distribution introduced by Birnbaum and Saunders (1969). It is often used to model lifetimes and reliability data, characterized by two parameters: shape and scale. Desmond (1986) highlighted an interesting aspect: the BirSau distribution can be formed by combining an inverse Gaussian (IG) distribution with its reciprocal. This relationship allows us to derive many characteristics of the BirSau distribution using the properties of the IG distribution. The BirSau distribution is highly versatile and has applications in various fields, especially in medical sciences. For example, Liu et al. (2023) utilized this distribution to analyze bone marrow transplant data. The readers interested in real-life applications of the BirSau distribution can check out the work by Balakrishnan and Kundu (2019) and its references therein. Jantakoon and Volodin (2019) proposed a novel approach to construct the confidence intervals of parameter of the BirSau distribution for both shape and scale parameters by applying two distinct methods. The numerous constructions, parameterizations, generalizations, and the inferential techniques which include different methods for parameter estimation had been done on this distribution have been developed or discussed by Ahmed et al. (2008), Santos-Neto et al. (2014).

Let $t_1, t_2, \ldots, t_n$ denote a random sample of positive values of size n that is drawn from a BirSau distribution with shape and scale parameters $(\alpha, \beta > 0)$ respectively. The probability density function of the BirSau distribution is defined by Lemonte et al. (2006) is given as

$$f\left(t;\alpha,\beta\right)=\frac{1}{2\alpha\beta\sqrt{2\pi}}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]\exp\left\{-\frac{1}{2\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right\}. \tag{1.1}$$

Over the past four decades, numerous studies have been published exploring different inferential methods for estimating the parameters of the BirSau distribution and their properties. Researchers have developed a keen interest in estimating these parameters, and this topic has recently attracted significant attention in the literature. The estimation approach based on modified moment estimators (MMEs) were provided by Ng et al. (2003) and showed through simulation results that the performance of MLEs and MMEs is nearly identical across different sample sizes, especially when the shape parameter α is not too large. While MLEs offer several appealing advantages, they often lack explicit expressions and can be quite sensitive to deviations from the model, which is a common issue in real-world situations. For this reason, it is important to consider alternative estimators to achieve more reliable estimation outcomes. Therefore, Wang et al. (2015) suggested an estimation methodology on some robust type of estimators with their asymptotic distribution.

Let $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta})^{'}$ be the parameter vector of the BirSau distribution and its robust estimators be denoted by $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}})^{'}$. We designated $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ as unrestricted robust (UR) estimators of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. According to the Wang et al. (2015), the robust estimators for both parameters are given as

$$\hat{\alpha}^{UR}=\frac{IQR\left(Y\right)}{1.34898}, \tag{1.2}$$

here, $\log(T) = Y$, and the interquartile range (IQR) is range of the third and first quartiles of the sample data. The asymptotic distribution for α is presented as follows:

$$\sqrt{n}\left(\hat{\alpha}^{UR}-\alpha\right)\xrightarrow{D}N\left(0,c\alpha^{2}\right). \tag{1.3}$$

Here c is a constant, $c\approx\frac{2.48}{1.3492}$, which is approximately 1.363. The parameter β which represents the median of the BirSau distribution is a logical choice. They refer to the sample median estimator of β as follows:

$$\hat{\beta}^{UR}=median\left(t_1,t_2,\ldots,t_n\right), \tag{1.4}$$

the asymptotic distribution $\hat{\beta}^{UR}$ is given as

$$\sqrt{n}\left(\hat{\beta}^{UR}-\beta\right)\xrightarrow{D}N\left(0,\frac{\pi\left(\alpha\beta\right)^{2}}{2}\right). \tag{1.5}$$

In many real-world statistical situations, some researchers are keen to blend sample data with non-sample information to make conclusions about population parameters. If the non sample is somewhat trustworthy then one can improve their inferences and achieve more efficient outcomes, particularly when the sample data is limited. On the other hand, if non sample information is unclear then some researchers often use some other efficient methods such as preliminary tests and James-Stein (shrinkage) techniques to reduce uncertainty. Saleh (1966) explored how to identify the best set of order statistics when estimating parameters of the exponential distribution, particularly when dealing with complete and censored samples. This work focuses on situations where the number of observations is significantly larger than a chosen integer k. The analysis is grounded in asymptotic theory and specifically addresses cases involving Type II censoring. Judge and Bock (1978) explored advanced statistical methods in econometrics, specifically focusing on pre-test estimators and Stein-rule estimators. Saleh (2006) provided the theory of preliminary test and Stein-type estimation with applications in a variety of standard models used in applied statistical inference. In past years, it has been become common practice to utilize non-sample information whenever it is available in the estimation process. For example, Shah et al. (2017) made inferences about the common mean problem of k samples which are drawn from a normal distribution. Similarly, Aldeni et al. (2023) using this approach as combining multiple p samples from log-normal populations with unequal variances, focusing on the calculation of log-normal distribution. Makhdoom et al. (2024a) did improved estimation for BirSau distribution for α while keeping other parameter known. Makhdoom et al. (2024b) considered the robust shape parameter of BirSau distribution and did improved estimation of BirSau distribution. This work is the extension of Makhdoom et al. (2024b) from single sample to multiple samples for both parameters.

The null hypothesis of homogeneity of α_i parameter can be used to present a constraint on the parametric space.

$$H_0:\alpha_1=\alpha_2=\cdots=\alpha_k=\alpha_0, \tag{1.6}$$

where α_0 is unknown. When the assumption of parameter homogeneity holds true, using a restricted model leads to estimates that perform statistically better than those based solely on the α estimator. However, if there are deviations from this null hypothesis, it can slightly impact the estimators derived from partial or combined estimates. To address this, we proposed improved estimation methods that incorporate both sample and non-sample data for the parametric vector for both parameters. In this current study, we kept the scale parameter β known while used improved estimation techniques to boost the efficiency of the shape parameter. One can replicate same estimation techniques for scale parameter β.

This paper is structured as follows: In Section 2, we outline various estimation methodologies designed to improve the inference of the both parameters. Section 3 focuses on the asymptotic distributional results and properties, including the mathematical formulation of the asymptotic risk for the proposed estimators. In Section 4, we present our simulation scheme along with the results, featuring graphical representations of the estimators to assess their behavior across different parameter levels and configurations. This section also includes a real-life data example of the failure times of machine valves. Finally, one can find the conclusions of this study in Section 5.

2 Estimation Strategies

To effectively estimate using both sample and non-sample information in optimal ways, we explore the following estimation methodologies:

2.1 Robust restricted (RR) estimator

The restricted estimator for α is given as

$$\hat{\boldsymbol{\alpha}}^{RR} = (\hat{\alpha}_G, \hat{\alpha}_G, \ldots, \hat{\alpha}_G)^{'} = \hat{\alpha}_G \mathbf{1}_k, \tag{2.1}$$

where $\hat{\alpha}_G$ is famous estimator of Graybill-Deal type Shah et al. (2017). It can be shown as

$$\hat{\alpha}_G = \frac{1}{\sum_{i=1}^{k} \frac{n_i}{\hat{V}_i}} \sum_{i=1}^{k} \frac{n_i}{\hat{V}_i} \hat{\alpha}_i^{UR} = \frac{1}{\sum_{i=1}^{k} \frac{w_{i,n}}{\hat{V}_i}} \sum_{i=1}^{k} \frac{w_{i,n}}{\hat{V}_i} \hat{\alpha}_i^{UR}, \tag{2.2}$$

where $w_{i,n} = \frac{n_i}{n}$ and $\hat{V}_i = c\hat{\alpha}^2$. The equation (2.2) can be shown in matrix notation as

$$\hat{\boldsymbol{\alpha}}^{RR} = \hat{W}^{-1} \boldsymbol{P}_k \hat{\boldsymbol{\Omega}}_n^{-1} \hat{\boldsymbol{\alpha}}^{UR} = \boldsymbol{L_n} \hat{\boldsymbol{\alpha}}^{UR}, \tag{2.3}$$

where $\boldsymbol{L_n} = \hat{W}^{-1} \boldsymbol{P}_k \hat{\boldsymbol{\Omega}}_n^{-1}$, $\hat{W} = \sum_{i=1}^{k} \frac{w_{i,n}}{\hat{V}_i}$, $\hat{\boldsymbol{\Omega}}_n = diag\left(\frac{\hat{V}_1}{w_{i,n}}, \frac{\hat{V}_2}{w_{i,n}}, \ldots, \frac{\hat{V}_k}{w_{i,n}}\right)$, and $\boldsymbol{P}_k = \mathbf{1}_k \mathbf{1}_k^{'}$ is a $(k \times k)$ matrix of ones. Based on the consistency of $\hat{\alpha}_i$ and $\hat{V}_i$, it can be shown that $\hat{W} \xrightarrow{P} W$, while $\hat{\boldsymbol{\Omega}}_n \xrightarrow{P} \boldsymbol{\Omega} = diag\left(\frac{V_1}{w_i}, \frac{V_2}{w_i}, \ldots, \frac{V_k}{w_i}\right)$, eventually $\boldsymbol{L_n} \xrightarrow{P} \boldsymbol{L}_0 = W^{-1} \boldsymbol{P}_k \boldsymbol{\Omega}^{-1}$, where $\lim_{n\to\infty} w_{i,n} = w_i$, $(0 < w_i < 1)$ is fixed for $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} w_i = 1$.

2.2 Robust linear shrinkage (RLS) estimator

The robust linear shrinkage (RLS) estimator $\hat{\alpha}^{RLS}$ is given as by after making linear combination of $\hat{\alpha}^{RR}$ and $\hat{\alpha}^{UR}$,

$$\hat{\boldsymbol{\alpha}}^{RLS} = \delta \hat{\boldsymbol{\alpha}}^{RR} + (1 - \delta) \hat{\boldsymbol{\alpha}}^{UR} = \hat{\boldsymbol{\alpha}}^{UR} - \delta \left(\hat{\boldsymbol{\alpha}}^{UR} - \hat{\boldsymbol{\alpha}}^{RR}\right). \tag{2.4}$$

In this context, $0 < \delta < 1$ represents the shrinkage coefficient intensity.

2.3 Robust preliminary test (RPT) estimator

When prior knowledge is somewhat uncertain, using a pretest estimator, denoted as $\hat{\boldsymbol{\alpha}}^{RPT}$, can be beneficial as it incorporates a test of H_0. Both James-Stein and pretest estimators depend on a test statistic for their construction. To test the null hypothesis (1.6), we can derive a large sample test statistic by calculating the normalized distance between $\hat{\boldsymbol{\alpha}}^{RRE}$ and $\hat{\boldsymbol{\alpha}}^{UR}$.

$$\mathscr{B}_n = \sqrt{n}\left(\hat{\boldsymbol{\alpha}}^{UR} - \hat{\boldsymbol{\alpha}}^{RR}\right)' \hat{\boldsymbol{\Omega}}_n^{-1}\left(\hat{\boldsymbol{\alpha}}^{UR} - \hat{\boldsymbol{\alpha}}^{RR}\right). \tag{2.5}$$

Under H_0, The sampling distribution $\mathscr{B}_n$ approaches a central χ^2 distribution with $k-1$ degrees of freedom. Consequently, we can use this χ^2 distribution to approximate the higher α^* level of significance values of $\mathscr{B}_n$, which are represented as $\mathscr{B}_{n,\alpha^*}$. The pretest (PT) estimator of $\boldsymbol{\alpha}$ is defined accordingly,

$$\hat{\boldsymbol{\alpha}}^{RPT} = \hat{\boldsymbol{\alpha}}^{RR} I\left(\mathscr{B}_n < \mathscr{B}_{n,\alpha^*}\right) + \hat{\boldsymbol{\alpha}}^{UR} I\left(\mathscr{B}_n > \mathscr{B}_{n,\alpha^*}\right),$$

where $I(\cdot)$ is an indicator function. For the computational purposes, the following form of the estimator can also be used as

$$\hat{\boldsymbol{\alpha}}^{RPT} = \hat{\boldsymbol{\alpha}}^{UR} - \left(\hat{\boldsymbol{\alpha}}^{UR} - \hat{\boldsymbol{\alpha}}^{RR}\right) I\left(\mathscr{B}_n < \mathscr{B}_{n,\alpha^*}\right). \tag{2.6}$$

2.4 Robust shrinkage preliminary test (RSP) estimator

The robust shrinkage preliminary test estimator of α is being derived after plugging $\hat{\boldsymbol{\alpha}}^{RR}$ with $\hat{\boldsymbol{\alpha}}^{RLS}$ by incorporating δ into (2.6). This leads us to the following expression for the RSP estimator.

$$\hat{\boldsymbol{\alpha}}^{RSP} = \hat{\boldsymbol{\alpha}}^{UR} - \delta\left(\hat{\boldsymbol{\alpha}}^{UR} - \hat{\boldsymbol{\alpha}}^{RR}\right) I\left(\mathscr{B}_n < \mathscr{B}_{n,\alpha^*}\right). \tag{2.7}$$

If $\delta = 0$ then $\hat{\boldsymbol{\alpha}}^{RSP}$ becomes $\hat{\boldsymbol{\alpha}}^{UR}$ and if $\delta = 1$ then $\hat{\boldsymbol{\alpha}}^{RSP}$ shrinkage to $\hat{\boldsymbol{\alpha}}^{LSR}$. It is evident that $\hat{\boldsymbol{\alpha}}^{RSP}$ outperforms $\hat{\boldsymbol{\alpha}}^{UR}$ across a broader range of the parametric space compared to $\hat{\boldsymbol{\alpha}}^{PT}$.

2.5 Robust James-Stein (RJS) estimator

The pretest estimators outlined in (2.6) are sensitive to deviate from H_0 and depend on the significance level α^*. To address this issue, we introduced the James Stein type estimator, denoted as $\hat{\boldsymbol{\alpha}}^{RJS}$, which is defined as follows

$$\hat{\boldsymbol{\alpha}}^{RJS} = \hat{\boldsymbol{\alpha}}^{UR} - \left\{(k-3)\,\mathscr{B}_n^{-1}\right\}\left(\hat{\boldsymbol{\alpha}}^{UR} - \hat{\boldsymbol{\alpha}}^{RR}\right) \qquad k \geq 4,$$

the above equation can be rephrased for easier understanding in computational contexts as follows

$$\hat{\boldsymbol{\alpha}}^{RJS} = \hat{\boldsymbol{\alpha}}^{RR} + \left\{1 - (k-3)\,\mathscr{B}_n^{-1}\right\}\left(\hat{\boldsymbol{\alpha}}^{UR} - \hat{\boldsymbol{\alpha}}^{RR}\right) \qquad k \geq 4. \tag{2.8}$$

2.6 Robust positive James-Stein (RPJ) estimator

The positive past of James-Stein estimator can be presented as,

$$\hat{\boldsymbol{\alpha}}^{RPJ} = \hat{\boldsymbol{\alpha}}^{RR} + max\{1 - (k-3)\,\mathscr{B}_n^{-1}, 0\}\left(\hat{\boldsymbol{\alpha}}^{UR} - \hat{\boldsymbol{\alpha}}^{RR}\right) \qquad k \geq 4,$$

alternatively, we can derive another computational form for the pretest James-Stein (PJS) estimator as,

$$\hat{\boldsymbol{\alpha}}^{RPJ} = \hat{\boldsymbol{\alpha}}^{RJS} - \{1 - (k-3)\,\mathscr{B}_n^{-1}\} I\left(\mathscr{B}_n < k-3\right)\left(\hat{\boldsymbol{\alpha}}^{UR} - \hat{\boldsymbol{\alpha}}^{RR}\right) \qquad k \geq 4. \tag{2.9}$$

3 Asymptotic Mathematical Results

This section will explore asymptotic properties of suggested estimators, focusing on following weighted squared error loss:

$$q\left(\hat{\boldsymbol{\alpha}}^{*}, \boldsymbol{\alpha}\right) = \sqrt{n}\left(\hat{\boldsymbol{\alpha}}^{*} - \boldsymbol{\alpha}\right)^{'} \boldsymbol{M}\sqrt{n}\left(\hat{\boldsymbol{\alpha}}^{*} - \boldsymbol{\alpha}\right), \tag{3.1}$$

where $\hat{\boldsymbol{\alpha}}^{*}$ represents any estimator of $\boldsymbol{\alpha}$, and $\boldsymbol{M}$ is a positive semi-definite matrix. The squared error risk is defined as follows:

$$\mathscr{R}\left(\hat{\boldsymbol{\alpha}}^{*}, \boldsymbol{\alpha}\right) = E\left[\sqrt{n}\left(\hat{\boldsymbol{\alpha}}^{*} - \boldsymbol{\alpha}\right)^{'} \boldsymbol{M}\sqrt{n}\left(\hat{\boldsymbol{\alpha}}^{*} - \boldsymbol{\alpha}\right)\right]. \tag{3.2}$$

As noted in (2.5), the test statistic remains consistent for a fixed $\boldsymbol{\alpha}$ under the null hypothesis H_0 against fixed alternatives. This indicates that all estimators based on $\mathscr{B}_n$ behave similarly to the unrestricted estimator as the sample size grows, providing limited scope for further exploration. To explore deeply, we will investigate a sequence of local alternatives to establish the asymptotic results.

$$\mathscr{K}_n : \boldsymbol{\alpha} = \boldsymbol{\alpha}_{(n)} = \boldsymbol{\alpha}_0 + \frac{\boldsymbol{\xi}}{\sqrt{n}}, \tag{3.3}$$

as $\boldsymbol{\xi} \in \mathbb{R}^k$ is a fixed real vector and $\boldsymbol{\alpha}_0 = \alpha_0 \mathbf{1}_k$. When $\boldsymbol{\xi} = 0$, we find that $\boldsymbol{\alpha}_{(n)} = \alpha_0 \mathbf{1}_k$. This indicates that (1.6) is a specific case of $\mathscr{K}_n$. For analysis we are introducing some important lemmas that will support our findings.

3.1 Lemmas

Lemma 1: Let $\boldsymbol{S} = \sqrt{n}\left(\hat{\boldsymbol{\alpha}}^{UR} - \boldsymbol{\alpha}_0\right)$, $\boldsymbol{X} = \sqrt{n}\left(\hat{\boldsymbol{\alpha}}^{UR} - \boldsymbol{\alpha}^{RR}\right)$ *and* $\boldsymbol{Y} = \sqrt{n}\left(\hat{\boldsymbol{\alpha}}^{RR} - \boldsymbol{\alpha}_0\right)$. *Thus, due to the sequence of local alternatives provided in (3.3), as* $n \to \infty$,

$$\begin{pmatrix} \boldsymbol{S}_n \\ \boldsymbol{X}_n \end{pmatrix} \xrightarrow{D} \begin{pmatrix} \boldsymbol{S} \\ \boldsymbol{X} \end{pmatrix} \sim \mathscr{N}_{2k}\left\{\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\xi}^{*} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Omega} & \boldsymbol{E}_0 \\ \boldsymbol{E}_0^{'} & \boldsymbol{E}_0 \end{pmatrix}\right\}, \tag{3.4}$$

$$\begin{pmatrix} \boldsymbol{Y}_n \\ \boldsymbol{X}_n \end{pmatrix} \xrightarrow{D} \begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{X} \end{pmatrix} \sim \mathscr{N}_{2k}\left\{\begin{pmatrix} \mathbf{0} \\ \boldsymbol{\xi}^{*} \end{pmatrix}, \begin{pmatrix} W^{-1}\boldsymbol{P} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{E}_0 \end{pmatrix}\right\}, \tag{3.5}$$

where $\boldsymbol{\xi}^* = \boldsymbol{C}_0\boldsymbol{\xi}$, $\boldsymbol{C}_0 = \boldsymbol{I}_k - \boldsymbol{G}_0$ and $\boldsymbol{E}_0 = \boldsymbol{\Omega C}_0^{'}$. Additionally, $\boldsymbol{W}^{*'}\boldsymbol{\xi} = 0$, where $\boldsymbol{W}^* = \left(\frac{w_1}{V_1}, \frac{w_2}{V_2}, \cdots \frac{w_k}{V_k}\right)^{'}$. The relations which are stated above can be extracted from Shah et al. (2017).

Lemma 2: If $n \to \infty$, test statistic $\mathscr{B}_n$ converges to a non-central chi-square distribution. This distribution has $k-1$ degrees of freedom and a non-centrality parameter denoted as $\Delta = \xi^*\Omega\xi^*$.

For an estimator $\hat{\boldsymbol{\alpha}}^*$ of the parameter $\boldsymbol{\alpha}$, the asymptotic distribution of the quantity $\sqrt{n}(\hat{\boldsymbol{\alpha}}^* - \boldsymbol{\alpha})$ under the $\mathscr{K}_n$ is described as follows

$$F(\boldsymbol{x}) = \lim_{n\to\infty} \mathbb{P}\left[\sqrt{n}\left(\hat{\boldsymbol{\alpha}}^* - \boldsymbol{\alpha}\right) \leq \boldsymbol{x} | \mathscr{K}_n\right], \tag{3.6}$$

If the limit exists, the asymptotic distributional bias (ADB) of an estimator $\hat{\boldsymbol{\alpha}}^*$ in relation to $F(\boldsymbol{x})$ is defined as follows.

$$b\left(\hat{\boldsymbol{\alpha}}^*\right) = \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\boldsymbol{\alpha}}^* - \boldsymbol{\alpha}\right)\right] = \int \cdots \int \boldsymbol{x} dF(\boldsymbol{x}). \tag{3.7}$$

In order to compare the bias of different estimators effectively, we apply a quadratic transformation to the bias defined in (3.7). This introduces the concept of asymptotic distributional quadratic bias (AQDB), which allows us to evaluate the performance of these estimators in a more insightful manner.

$$aqb\left(\hat{\boldsymbol{\alpha}}^*\right) = \left[b\left(\hat{\boldsymbol{\alpha}}^*\right)\right]^{'} \boldsymbol{\Omega}^{-1}\left[b\left(\hat{\boldsymbol{\alpha}}^*\right)\right]. \tag{3.8}$$

In general, the estimators that utilize preliminary testing and shrinkage techniques often exhibit bias. In next subsection, we will discuss the asymptotic quadratic biases linked to the proposed estimators.

3.2 Asymptotic biases

Theorem 1. *In the context of local alternatives and the standard conditions required for Robust estimation, we have*

$$\begin{aligned}
aqb\left(\hat{\boldsymbol{\alpha}}^{RR}\right) &= \Delta,\\
aqb\left(\hat{\boldsymbol{\alpha}}^{RLS}\right) &= \delta^2\Delta,\\
aqb\left(\hat{\boldsymbol{\alpha}}^{RPT}\right) &= \Delta\left[G_{k+1}\left(\chi^2_{k-1,\alpha^*};\Delta\right)\right]^2\\
aqb\left(\hat{\boldsymbol{\alpha}}^{RSP}\right) &= \delta^2\Delta\left[G_{k+1}\left(\chi^2_{k-1,\alpha^*};\Delta\right)\right]^2\\
aqb\left(\hat{\boldsymbol{\alpha}}^{RJS}\right) &= (k-3)^2\Delta\left[E\left(\chi^{-2}_{k+1}(\Delta)\right)\right]^2\\
aqb\left(\hat{\boldsymbol{\alpha}}^{RPJ}\right) &= \Delta\left[G_{k+1}(k-3;\Delta) + (k-3)E\left[\chi^{-2}_{k+1}(\Delta) I\left(\chi^{-2}_{k+1}(\Delta) > (k-3)\right)\right]\right]^2,
\end{aligned}$$

where $G_\nu(\cdot;\Delta)$ represents the cumulative distribution function (CDF) of a non-central chi-square distribution, which has ν degrees of freedom and a non-centrality parameter Δ.

Proof. The detail proofs of these relation can be found in the similar way as in Shah et al. (2017) and Aldeni et al. (2023). □

3.3 Asymptotic distributional quadratic risk (ADQR)

According to Ahmed (2014), the asymptotic distributional quadratic risk (ADQR) of an estimator illustrates the behavior of estimation error as the sample size grows. The ADQR for an estimator $\hat{\boldsymbol{\alpha}^*}$ can be expressed as follows:

$$\mathscr{R}\left(\hat{\boldsymbol{\alpha}^*}, \boldsymbol{M}\right) = tr\left[\boldsymbol{M}\boldsymbol{\Sigma}\left(\hat{\boldsymbol{\alpha}^*}\right)\right], \tag{3.9}$$

where $\boldsymbol{\Sigma}\left(\hat{\boldsymbol{\alpha}^*}\right)$ denotes the asymptotic mean-squared error matrix (AMSEM) of $\hat{\boldsymbol{\alpha}^*}$. This matrix is defined as follows::

$$\boldsymbol{\Sigma}\left(\hat{\boldsymbol{\alpha}^*}\right) = \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\boldsymbol{\alpha}}^* - \boldsymbol{\alpha}\right)\sqrt{n}\left(\hat{\boldsymbol{\alpha}}^* - \boldsymbol{\alpha}\right)^{'}\right] = \int\cdots\int \boldsymbol{x}\boldsymbol{x}^{'} dF\left(\boldsymbol{x}\right). \tag{3.10}$$

Now, we examine the asymptotic distributional quadratic risk (ADQR) for the proposed estimators outlined in the following theorem.

Theorem 2. *The expression for the ADQR of the estimators, when we take into account the sequence of local alternatives with $\boldsymbol{M} = \boldsymbol{\Sigma}^{-1}$, is given as follows*

$$\begin{aligned}
R\left(\hat{\boldsymbol{\alpha}}^{UR};\boldsymbol{\Sigma}^{-1}\right) &= k,\\
R\left(\hat{\boldsymbol{\alpha}}^{RR};\boldsymbol{\Sigma}^{-1}\right) &= 1+\Delta,\\
R\left(\hat{\boldsymbol{\alpha}}^{RLS};\boldsymbol{\Sigma}^{-1}\right) &= k-\delta\left(2-\delta\right)\left(k-1\right)+\delta^2\Delta,\\
R\left(\hat{\boldsymbol{\alpha}}^{PTR};\boldsymbol{\Sigma}^{-1}\right) &= k-\left(k-1\right)G_{p+1}\left(\chi^2_{k-1,\alpha^*};\Delta\right)\\
&\quad+\Delta\left[2G_{k+1}\left(\chi^2_{k-1,\alpha^*};\Delta\right)-G_{k+3}\left(\chi^2_{k+1,\alpha^*};\Delta\right)\right]\\
R\left(\hat{\boldsymbol{\alpha}}^{RSP};\boldsymbol{\Sigma}^{-1}\right) &= k-\delta\left(2-\delta\right)\left(k-1\right)G_{k+1}\left(\chi^2_{k-1,\alpha^*};\Delta\right)\\
&\quad+\delta\Delta\left[2G_{k+1}\left(\chi^2_{k-1,\alpha^*};\Delta\right)-\left(2-\delta\right)G_{k+3}\left(\chi^2_{k-1,\alpha^*};\Delta\right)\right]\\
R\left(\hat{\boldsymbol{\alpha}}^{RJS};\boldsymbol{\Sigma}^{-1}\right) &= k-\left(k-1\right)\left(k-3\right)\left\{2E\left[\chi^{-2}_{k+1}\left(\Delta\right)\right]-\left(k-3\right)E\left[\chi^{-4}_{k+1}\left(\Delta\right)\right]\right\}\\
&\quad+\left(k-3\right)\left(k+1\right)\Delta E\left[\chi^{-4}_{k+3}\left(\Delta\right)\right]\\
R\left(\hat{\boldsymbol{\alpha}}^{RPJ};\boldsymbol{\Sigma}^{-1}\right) &= R\left(\hat{\boldsymbol{\alpha}}^{RJS};\boldsymbol{\Sigma}^{-1}\right)-\left(k-1\right)E\left\{\left[1-\left(k-3\right)\chi^{-2}_{k+1}\left(\Delta\right)\right]^2 I\left(\chi^2_{k+1}\left(\Psi\right)<\left(k-3\right)\right)\right\}\\
&\quad+\Delta\left[2E\left\{\left[1-\left(k-3\right)\chi^{-2}_{k+1}\left(\Delta\right)\right]I\left(\chi^2_{k+1}\left(\Delta\right)<\left(k-3\right)\right)\right\}\right]\\
&\quad-E\left\{\left[1-\left(k-3\right)\chi^{2}_{k+1}\left(\Delta\right)\right]^2 I\left(\chi^2_{k+1}\left(\Delta\right)<\left(k-3\right)\right)\right\}.
\end{aligned}$$

Proof. For detailed proof see the methodologies of Shah et al. (2017) and Aldeni et al. (2023). □

4 Numerical Evaluation

In this section, the performance of the suggested estimators is presented through simulation and real life data application.

4.1 Simulation

The simulated data is generated through *R* programming language utilizing *rbs* function from the *bsgof* package with different sample sizes i.e., $n_i = 30, 50,$ and 100 for multiple samples $k = 4, 6, 8$ and 10. We set $\alpha = 1$ and $\beta = 1$, following the earlier work by Makhdoom et al. (2024a). The entire simulation scheme replicated $R^\star = 10,000$. We calculated simulated relative efficiency (SRE) to assess the performance of recommended estimators. The SRE is being defined here as

$$SRE\left(\hat{\boldsymbol{\alpha}}^{UR} : \hat{\boldsymbol{\alpha}}^{*}\right) = \frac{SR\left(\hat{\boldsymbol{\alpha}}^{UR}\right)}{SR\left(\hat{\boldsymbol{\alpha}}^{*}\right)},$$

where $\hat{\boldsymbol{\alpha}}^{*}$ is any of the recommended estimator in section of improved estimation strategies. If the simulated relative efficiency (SRE) is greater than 1, it indicates that $\hat{\boldsymbol{\alpha}}^{*}$ is more efficient than $\hat{\boldsymbol{\alpha}}^{UR}$ and same for scale parameter. We also introduced a shift parameter $\lambda = (\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)^{'} (\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)$ which measures the distance from the assumed non-sample information (NSI) $\boldsymbol{\alpha}_0$. In our simulation experiments, we tested different shrinkage intensity factor $\delta = 0.25, 0.50, 0.75$ and significance levels $\alpha^* = 0.01, 0.05, 0.10$. We examined a wide variety of combinations for n_i, k, δ, and α^*. However, to keep the presentation straightforward, we have focused on the results for $\alpha^* = 0.01, 0.05$ with $k = 4, 6$ and $\delta = 0.50$. The performance of the suggested estimators in terms of SRE can be found in the following tables.

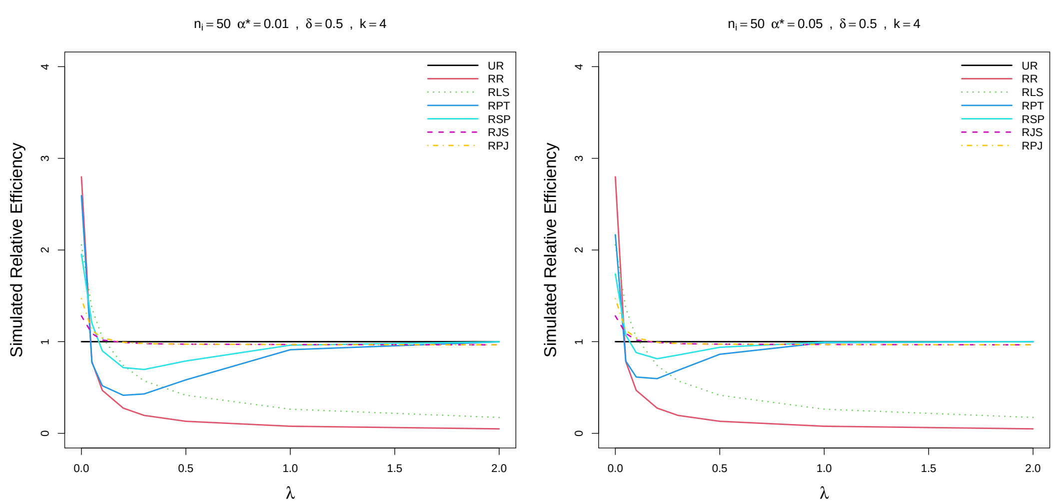

Figure 1: Simulated Relative Efficiency of the estimators $k = 4$

Table 1: SREs of the estimators relative to $\hat{\boldsymbol{\alpha}}^{UR}$ for $\delta = 0.5, \alpha = 1, \beta = 1, n_i = 30, 50, 100$, $\boldsymbol{k = 4}$

				$\hat{\boldsymbol{\alpha}}^{RPT}$		$\hat{\boldsymbol{\alpha}}^{RSP}$			
n_i	λ	$\hat{\boldsymbol{\alpha}}^{RR}$	$\hat{\boldsymbol{\alpha}}^{RLS}$	$\alpha^* = 0.01$	$\alpha^* = 0.05$	$\alpha^* = 0.01$	$\alpha^* = 0.05$	$\hat{\boldsymbol{\alpha}}^{RJS}$	$\hat{\boldsymbol{\alpha}}^{RPJ}$
50	0.00	2.8125	2.0579	2.6259	2.1718	1.9671	1.7407	1.2981	1.4751
	0.05	0.7787	1.3615	0.7697	0.7871	1.1922	1.0684	1.0884	1.1256
	0.10	0.4721	1.3615	0.5194	0.6160	0.8981	0.8792	1.0130	1.0355
	0.20	0.2753	0.7428	0.4137	0.6074	0.7167	0.8202	0.9911	0.9930
	0.30	0.1968	0.5781	0.4329	0.6919	0.6997	0.8568	0.9836	0.9838
	0.50	0.1291	0.4116	0.5852	0.8547	0.7901	0.9348	0.9735	0.9735
	1.00	0.0777	0.2638	0.9201	0.9933	0.9649	0.9970	0.9673	0.9673
	2.00	0.0481	0.1714	1.0000	1.0000	1.0000	1.0000	0.9669	0.9669
	3.00	0.0380	0.1376	1.0000	1.0000	1.0000	1.0000	0.9670	0.9670
	4.00	0.0320	0.1163	1.0000	1.0000	1.0000	1.0000	0.9631	0.9631
100	0.00	3.7139	2.2699	3.2851	2.5810	2.1340	1.8751	1.3495	1.5365
	0.05	0.2708	0.7668	0.4564	0.6524	0.7452	0.8460	1.0136	1.0144
	0.10	0.1476	0.4823	0.6283	0.8740	0.8221	0.9454	1.0005	1.0005
	0.20	0.1043	0.3622	0.8512	0.9675	0.9353	0.9861	1.0008	1.0008
	0.30	0.0631	0.2294	0.9955	0.9984	0.9981	0.9993	0.9944	0.9944
	0.50	0.0489	0.1823	0.9993	0.9993	0.9997	0.9997	0.9962	0.9962
	1.00	0.0385	0.1449	1.0000	1.0000	1.0000	1.0000	0.9936	0.9936
	2.00	0.0238	0.0916	1.0000	1.0000	1.0000	1.0000	0.9945	0.9945
	3.00	0.0187	0.0727	1.0000	1.0000	1.0000	1.0000	0.9946	0.9946
	4.00	0.0160	0.0627	1.0000	1.0000	1.0000	1.0000	0.9993	0.9993

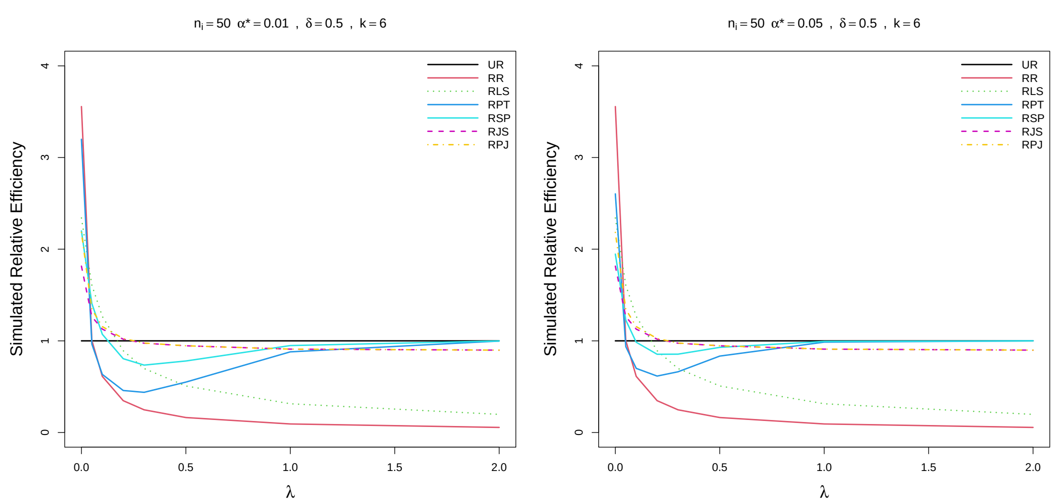

Figure 2: Simulated Relative Efficiency of the estimators for $k = 6$

Table 2: SREs of the estimators relative to $\hat{\boldsymbol{\alpha}}^{UR}$ for $\delta = 0.5, \alpha = 1, \beta = 1$ $n_i = 30, 50, 100$, $\boldsymbol{k = 6}$

				$\hat{\boldsymbol{\alpha}}^{RPT}$		$\hat{\boldsymbol{\alpha}}^{RSP}$			
n_i	λ	$\hat{\boldsymbol{\alpha}}^{RR}$	$\hat{\boldsymbol{\alpha}}^{RLS}$	$\alpha^* = 0.01$	$\alpha^* = 0.050$	$\alpha^* = 0.01$	$\alpha^* = 0.05$	$\hat{\boldsymbol{\alpha}}^{RJS}$	$\hat{\boldsymbol{\alpha}}^{RPJ}$
50	0.00	3.5560	2.3406	3.1989	2.6038	2.1995	1.9467	1.8151	2.1833
	0.05	1.0044	1.6080	0.9647	0.9373	1.4071	1.2266	1.2637	1.3448
	0.10	0.6124	1.2664	0.6340	0.6997	1.0719	0.9850	1.1280	1.1555
	0.20	0.3476	0.8926	0.4584	0.6157	0.8054	0.8537	1.0185	1.0255
	0.30	0.2478	0.6983	0.4384	0.6625	0.7353	0.8548	0.9746	0.9763
	0.50	0.1634	0.5078	0.5499	0.8339	0.7802	0.9290	0.9460	0.9463
	1.00	0.0933	0.3137	0.8800	0.9860	0.9486	0.9940	0.9097	0.9097
	2.00	0.0556	0.1978	0.9978	1.0000	0.9991	1.0000	0.8987	0.8987
	3.00	0.0436	0.1575	1.0000	1.0000	1.0000	1.0000	0.8914	0.8914
	4.00	0.0363	0.1316	1.0000	1.0000	1.0000	1.0000	0.8761	0.8761
100	0.00	4.4373	2.4870	3.9074	3.01322	2.3442	2.0564	1.8915	2.3229
	0.05	0.6297	1.3295	0.6484	0.7156	1.0664	0.9846	1.1517	1.1784
	0.10	0.3492	0.9179	0.4990	0.6784	0.8187	0.8813	1.0525	1.0559
	0.20	0.1894	0.5866	0.6209	0.8591	0.8293	0.9420	0.9955	0.9955
	0.30	0.1327	0.4384	0.8231	0.9598	0.9250	0.9834	0.9799	0.9799
	0.50	0.0844	0.2939	0.9826	1.0000	0.9928	1.0000	0.9574	0.9574
	1.00	0.0482	0.1763	1.0000	1.0000	1.0000	1.0000	0.9474	0.9474
	2.00	0.0288	0.1078	1.0000	1.0000	1.0000	1.0000	0.9359	0.9359
	3.00	0.0214	0.0813	1.0000	1.0000	1.0000	1.0000	0.9349	0.9349
	4.00	0.0178	0.0683	1.0000	1.0000	1.0000	1.0000	0.9356	0.9356

- The performance of the $\hat{\boldsymbol{\alpha}}^{RR}$ estimator is outstanding as the sample size and the number of variables k increase particularly when λ is exactly zero. However, as λ moves away from zero, the Simulated Relative Efficiency (SRE) of $\hat{\boldsymbol{\alpha}}^{RR}$ decreases significantly, approaching zero for all positive values of λ. In simpler terms, the $\hat{\boldsymbol{\alpha}}^{RR}$ estimator performs exceptionally well when the shift parameter is exactly zero, but its effectiveness declines rapidly as the shift parameter deviates from this point.

- The Simulated Relative Efficiency (SRE) for the $\hat{\boldsymbol{\alpha}}^{RLS}$ estimator also approaches zero, but at a slower rate compared to $\hat{\boldsymbol{\alpha}}^{RR}$. The linear shrinkage estimator performs better and is more comparable in terms of relative efficiency with the other estimators, particularly for small values of shrinkage intensity.

- The $\hat{\boldsymbol{\alpha}}^{RPT}$ is better performer relative to $\hat{\boldsymbol{\alpha}}^{RSP}$ for $\lambda = 0$. As we move, $\lambda > 0$ the $\hat{\boldsymbol{\alpha}}^{RSP}$ is better for all values of λ relative to $\hat{\boldsymbol{\alpha}}^{RPT}$. The $\hat{\boldsymbol{\alpha}}^{RPT}$ is best as compared to $\hat{\boldsymbol{\alpha}}^{RSP}$ for small region, not uniformly. The SRE of pretest estimator and shrinkage preliminary test estimator for some points become inferior to $\hat{\boldsymbol{\alpha}}^{URR}$, consequently SRE reaches to 1 for both $k = 4, 8$ for all sample sizes.

- The performance of the $\hat{\boldsymbol{\alpha}}^{RJS}$ and $\hat{\boldsymbol{\alpha}}^{RPJ}$ estimators is superior compared to the benchmark $\hat{\boldsymbol{\alpha}}^{UR}$ estimator for small values of λ. For the higher values of λ, both James Stein estimators become vulnerable as compared to $\hat{\boldsymbol{\alpha}}^{UR}$.

4.2 Real life application

The real life data example about the failure times is analyzed in this subsection using bootstrapping re-sampling methods.

Machine valves

This dataset contains machine valve failure times from an industrial process, organized into $n_i = 30$ samples, each with $k = 5$. We are analyzing this data, taken from Leiva (2015). In order to explore ways to account for the shape parameter α and improve the estimation of the BirSau distribution we apply our proposed improved estimation methodology to this data, generating bootstrap samples for the actual data sets. We analyzed the mentioned data using our proposed enhanced estimation methodology, generating bootstrap samples from the actual data set.

- We carried out an experiment by starting with an initial sample of 50 observations, referred to as $n_0 = 50$. We then repeated this process $10,000$ times to generate multiple data sets.

- For each of bootstrap samples, we computed an unrestricted estimator along with a test statistic known as B_n.

- We calculated the RLS, the RPT estimator and RSP estimator after fixing the shrinkage intensity $\delta = 0.5$.

- Finally, we calculated the SRE for each of these three estimators across all bootstrap samples. The SRE serves as a measure to help us understand how well each estimator performs in comparison to the others.

- In summary, we performed a thorough statistical analysis that included re-sampling techniques, parameter estimation, and performance evaluation to extract meaningful insights from the real data.

The estimators $\hat{\boldsymbol{\alpha}}^{RJS}$ and $\hat{\boldsymbol{\alpha}}^{RPJ}$ are more efficient than their competitors, exhibiting the highest relative efficiency. Our simulations showed that $\hat{\boldsymbol{\alpha}}^{RSP}$ outperformed $\hat{\boldsymbol{\alpha}}^{RPT}$ at the same significance

Table 3: Simulated relative efficiencies of the estimators relative to $\hat{\boldsymbol{\alpha}}^{UR} = 0.3005$

				$\hat{\boldsymbol{\alpha}}^{RPT}$		$\hat{\boldsymbol{\alpha}}^{RSP}$			
n_0	δ	$\hat{\boldsymbol{\alpha}}^{RRE}$	$\hat{\boldsymbol{\alpha}}^{RLS}$	$\alpha^* = 0.01$	$\alpha^* = 0.05$	$\alpha^* = 0.01$	$\alpha^* = 0.05$	$\hat{\boldsymbol{\alpha}}^{RJS}$	$\hat{\boldsymbol{\alpha}}^{RPJ}$
50	0.5	0.2851	0.9886	0.5368	0.7598	0.8845	0.9236	1.2587	1.6854

level α^* with a shrinkage intensity factor of $\delta = 0.5$. This example highlights how improved estimation methods for the shape parameter α can be beneficial when combining multiple samples from the BirSau distribution.

5 Conclusion

In this study, we implemented advanced estimation techniques, including pretest, shrinkage pretest, and Stein-type methods, across k populations in a large sample context for the shape parameter of the BirSau distribution. We introduced six improved estimation methodologies for the shape parameter BirSau distribution and compared their asymptotic properties with the unrestricted robust estimator. Additionally, we derived a large sample test statistic based on the normalized distance between the unrestricted and restricted estimators and performed Monte Carlo simulations to support our theoretical findings. Our results indicate that the restricted estimator outperformed other estimators for the true parameter value, particularly when the common mean hypothesis holds. The pretest estimators also showed strong performance, surpassing the restricted and Stein-type estimators in certain regions of the parameter space, although they eventually lagged behind the unrestricted estimator. Notably, Stein-type estimators demonstrated impressive performance, especially for small values of the shift parameter. The simulation results, along with graphical representations of the estimators' performance, validate our asymptotic theoretical conclusions. We recommend that if researchers are uncertain about non-sample information, they should consider using the positive-part Stein estimator, especially when the number of populations $k \geq 4$. For smaller dimensions ($k \leq 3$), the shrinkage pretest estimator is the preferred choice.

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