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Positivity Expansion to Signed Solution for Doubly Nonlinear Parabolic Equations

Research Article

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ABSTRACT

We investigate doubly nonlinear parabolic equation with sign changing solutions. We established the expansion of positivity to the sign changing solution within a parabolic domain which is the key elements to achieve the regularity results.

Keywords: *Expansion of positivity, Cylindrical domain, p-Laplacian, Parabolic equation*

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ and for $T > 0$ define the cylindrical domain $\Omega_T := \Omega \times (0, T]$. Consider the following doubly nonlinear parabolic equation

$$\partial_t(|u|^{p-2}u) - div(|Du|^{p-2}Du) = 0 \quad \text{weakly in } \Omega_T \quad (1)$$

where $\Delta_p u := \text{div}\,(|\text{Du}|^{\text{p}-2}\text{Du})$ is the p-Laplacian.

For the case $p = 2$ then this operator transforms to well known heat equation. In this manuscript, the weak solution u is unknown and assumed to be locally bounded, real function which depends on both the time and space variables namely x and t in the cylindrical domain.

In our context, the term structural data indicates the parameters p and N. It is also assumed that the constant γ_0 , need to be evaluated quantitatively apriori in terms of the structural data. In addition, denote $\Gamma_T := \partial\Omega_T - \bar{\Omega} \times \{T\}$ to be the parabolic boundary of the cylindrical domain Ω_T. For $\theta > 0$,consider the following backward cylinders of the form

$$\begin{aligned}(x_0, t_0) + Q_\varrho(\theta) &= (x_0, t_0) + K_\varrho(0) \times (-\theta\varrho^p, 0] \\ &= K_\varrho(x_0) \times (t_0 - \theta\varrho^p, t_0].\end{aligned}$$

For the case $\theta = 1$, we will call it as Q_ϱ. If we assume that

$$K_{8\varrho}(x_0) \times (t_0, t_0 + (8\varrho)^p \subset Q. \quad (2)$$

Then the inference to be drawn from this article's findings is outlined below.

Proposition 1.1 *Suppose that u be a locally bounded, local weak sub(super) solution to (1) in Ω_T , for some $(x_0, t_0) \in \Omega_T, M > 0, \alpha \in (0, 1)$ and $\varrho > 0$. If the hypotheses (2) and*

$$|\{\pm(\mu^\pm - u(., t_0)) \geq M\} \cap K_\varrho(x_0)| \geq \alpha |K_\varrho|$$

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holds. There are constants ξ, δ and η in (0,1) that depend only on the data and α, and are such that either

$$|\mu^{\pm}| > \xi M$$

or

$$\pm(\mu^{\pm} - u) \geq \eta M \ \ a.e. in K_{2\varrho}(x_0) \times (t_0 + \delta(\frac{1}{2}\varrho)^p, t_0 + \delta\varrho^p),$$

where, $\xi = 2\eta$ in the degenerate case and $\xi = 8$ in the singular case.

In Definition 3, the idea of a weak solution is given. Before deriving the estimates we need to be sure that our equation has weak solutions. The existence of global weak solution is established in (Kuusi et al., 2021, Nakamura et al., 2018).

1.1 Newness and Importance

The equation (1) is a standard equation and is known as Trudinger's equation. It is frequently alluded to as a doubly nonlinear parabolic equation since both the solution and its spatial gradient exhibit nonlinear behavior. Since this type of equation has a beautiful mathematical structure, produces mixed sorts of degeneracy and/or singularity in partial differential equations, and connects to mathematical models like glacier dynamics (Mahaffy, 1976), shallow water flows(Alanso et al., 2008, Feng, 1997, Hromadka et al., 1985), and friction-dominated movement in a gas network (Leugering et al., 2018), it is particularly interesting to understand why we are choosing it for research. The Trudinger equation is also naturally connected to the non-linear eigenvalue issue $-\Delta_p u = \lambda|u|^{p-2}u$ (Lindgren et al.,2022), which plays an crucial role in the nonlinear potential theory. The energy estimation of signed solutions studied by V. Bögelein, F. Duzzar and N. Liao in (Bogelion et al., 2021) for a more general equations with structure conditions. The energy inequalities of this equation is studied by Trudinger (Trudinger, 1968)], for non-negative weak solutions, identical to the heat equation. The same is analyzed for non-negative weak solutions in (Kuusi et al., 2012, Kuusi1 et al., 2012). The positivity estimation made in (Misawa, 2023) for the nonnegative weak solution to doubly nonlinear equation. Now its turn to mention our contribution here, we eradicate the restriction of non-negativity of solutions instead we choose sign changing solutions for the energy inequalities to hold.The existence of weak solution to (1) is shown in (Nakamura et al., 2018). Since the Harnack inequality (Giannaza, 2006, Urbano, 2008) is invalid in our context due to the sign-changing solutions, we must use expansion of positivity to analyze the regularity of solutions to doubly nonlinear equations. In our upcoming effort, we will build the Hölder regularity, which is entirely founded on positivity expansion. The energy inequalities and expansion of positivity for doubly nonlinear equations has also been studied in (Ivanov, 1995, 1994, 1991, Kinnunen et al., 2007, Sarkar, 2022, Vespri, 1992, 2020).

2 Preliminaries

We set up several technical analysis tools and notations that will be used later [cf. (Bogelion et al., 2021, DiBenedetto, 2016, 1993, 1986, 2012, 1983, Evans, 1983, Ladyzenskaja et al., 1968, Leugering et al., 2018)].

2.1 Symbolization

2.1.1 A Local Weak Solution Interpretation.

A function

$$u \in C(0,T; L^p_{loc}(\Omega)) \cap L^p_{loc}(0,T; W^{1,p}_{loc}(\Omega))$$

is a local weak sub(super)-solution to (1), if for every compact set $C \subset \Omega$ and every sub $-$ interval $[t_1, t_2] \subset (0,T]$

$$\int_C |u|^{p-2}u\zeta dx|_{t_1}^{t_2} + \iint_{C\times(t_1,t_2)} [-|u|^{p-2}u\zeta_t + |Du|^{p-2}Du.D\zeta]dxdt \leq (\geq) 0 \qquad (3)$$

for all non-negative test functions

$$\zeta \in W^{1,p}_{loc}(0,T; L^p(C)) \cap L^p_{loc}(0,T; W^{1,p}_0(C)).$$

which guarantees that each and every integral in (3) converges. A local weak solution is a function u that is both a local weak subsolution and a local weak super-solution to (3).

2.1.2 Function Spaces on a Time-space Area.

We create a number of function spaces that operate on space-time barriers. For $1 \leq p, q \leq \infty, L^q(t_1, t_2; L^p(\Omega))$ is a space of measurable real-valued functions on $\Omega \times (t_1, t_2)$, a finite-norm space-time area, with an unbounded norm

$$\| v \|_{L^q(t_1,t_2;L^p(\Omega))}: = \begin{cases} \left(\int_{t_1}^{t_2} \| v(t) \|_{L^p(\Omega)}^q \, dt \right)^{1/q} & (1 \leq q < \infty) \\ esssup_{t_1 \leq t \leq t_2} \| v(t) \|_{L^p(\Omega)} & (q = \infty), \end{cases}$$

where

$$\| v(t) \|_{L^p(\Omega)}: = \begin{cases} \left(\int_{\Omega} |v(x,t)|^p \, dx \right)^{1/p} & (1 \leq p < \infty) \\ ess\ sup_{x \in \Omega} |v(x,t)| & (p = \infty). \end{cases}$$

For sake of simplicity, we write $L^p(\Omega \times (t_1, t_2)) = L^p(t_1, t_2 \,; L^p(\Omega))$ when $p = q$. For $1 \leq p < \infty$ the space $W^{1,p}(\Omega)$, further known as, Sobolev Space, is made up of weakly differentiable measurable real-valued functions whose weak derivatives are p-th integrable on Ω, with the norm

$$\| w \|_{W^{1,p}(\Omega)}: = \left(\int_{\Omega} |w|^p + |\nabla w|^p \, dx \right)^{1/p},$$

where $\nabla w = (w_{x_1}, \dots, w_{x_n})$ indicates, in a distribution sense, the gradient of w, and let $W_0^{1,p}(\Omega)$ represent the closure of $C_0^\infty(\Omega)$ in conjunction with the norm $\|\cdot\|_{W^{1,p}}$. Additionally, let's define $L^q(t_1, t_2 \,; W_0^{1,p}(\Omega))$ as a function space of measurable real-valued functions on a space-time area with a confined norm.

$$\| w \|_{L^q(t_1,t_2 \,; W_0^{1,p}(\Omega))}: = \left(\int_{t_1}^{t_2} \| w(t) \|_{W^{1,p}(\Omega)}^q \ dt \right)^{1/q}.$$

Let the domain $\Omega \subset \mathbb{R}^n$ be bounded. We describe the truncation of a function v, for a real number m, via

$$(v - m)_+ \coloneqq \max\{(v - m), 0\};$$
$$(v - m)_-: = -\min\{(v - m), 0\}. \qquad (4)$$

For a measurable function v in $L^1(\Omega)$ and a couple of real numbers $m < n$, we put

$$\begin{cases} \Omega \cap \{v > n\} \coloneqq \{x \in \Omega : \ v(x) > n\} \\ \Omega \cap \{v < m\} \coloneqq \{x \in \Omega : \ v(x) < m\} \\ \Omega \cap \{m < v < n\} \coloneqq \{x \in \Omega : \ m < v(x) < n\}. \end{cases} \qquad (5)$$

2.2 Technical Tools

Following (DiBenedetto, 1993), we define the auxiliary function

$$\begin{cases} A^+(k,u): = +(p-1) \int_k^u |s|^{p-2}(s-k)_+ \, ds \\ A^-(k,u): = -\int_k^u |s|^{p-2}(s-k)_- \, ds \end{cases} \qquad (6)$$

for $u, k \in \mathbb{R}$. If $k = 0$, we abbreviate as

$A^+(u) = A^+(0,u) \quad and \quad A^-(u) = A^-(0,u)$. It is clear that $A^\pm \geqslant 0$. Now, we may use bold typeface $\boldsymbol{b}^\alpha$ to represent the signed α-exponent of b as below

$$\boldsymbol{b}^\alpha = \begin{cases} |b|^{\alpha-1} \, b, b \neq 0, \\ 0, \quad\ b = 0. \end{cases}$$

We state a known lemma; Acerbi and Fusco, 1989, Trudinger[a], 1968) Lemma 2.2 for $\alpha \in (0,1)$ and (Giaquinta and Modica, 2006) ,inequality (2.4) for $\alpha > 1$. It is used to prove the next lemma.

Lemma 2.1 1 *The constant $\beta = \beta(\alpha)$ exists for each $\alpha > 0$, such that the inequality stated below holds true for all $a, b \in R$:*

$$\frac{1}{\beta} |b^\alpha - a^\alpha| \leqslant (|a| + |b|)^{\alpha-1} |b - a| \leqslant \beta |b^\alpha - a^\alpha|.$$

On the basis of the above lemma, We establish what comes next.

Lemma 2.2 2 *There exists a constant $\beta = \beta(p)$ such that the following inequality is true for all $w, k \in R, \ \alpha > 0$*

$$\frac{1}{\beta} (|w| + |k|)^{p-2} (w-k)_\pm^2 \leqslant A^\pm(k,w) \leqslant \beta (|w| + |k|)^{p-2} (w-k)_\pm^2$$

Proof. We will try to exhibit the proof of A^- and the estimate for the case A^+ is analogous. For $k \leqslant w$, we have $A^-(k,w) = 0 = (w-k)_-$. Therefore, we will consider only $k, w \in \mathbb{R}$ such that $w < k$. Then, we have

$$A^-(k,w) = (p-1) \int_w^k |s|^{p-2}(k-s) \, ds$$

$$\geqslant (p-1)\int_w^{\frac{1}{2}(k+w)} |s|^{p-2}(k-s)\,ds$$
$$\geqslant \frac{p-1}{2}(k-w)\int_w^{\frac{1}{2}(k+w)} |s|^{p-2}\,ds$$

Since $p-2>-1$ and Consequently, the integral on the right side is valid. When we use the previous lemma, we obtain

$$A^-(k,w) \geqslant \frac{1}{2}(k-w)|s|^{p-2}s|_w^{\frac{1}{2}(k+w)}$$
$$\geqslant \frac{1}{\beta(p)}(k-w)(|w|+\frac{1}{2}|k+w|)^{p-2}(\frac{1}{2}(k+w)-w)$$
$$=\frac{1}{2\beta(p)}(k-w)^2(|w|+\frac{1}{2}|k+w|)^{p-2}$$
$$\geqslant \frac{1}{\beta(p)}(k-w)^2(|w|+|k|)^{p-2}$$

In our last computation, we have used the reasoning $\frac{1}{2}(|k|+|w|) \leqslant |w|+\frac{1}{2}|k+w| \leqslant 2(|w|+|k|)$. This is the lower bound on A^-. Using the same lemma we can obtain

$$A^-(k,w) = (p-1)\int_w^k |s|^{p-2}(k-s)\,ds$$
$$\leqslant (p-1)(k-w)\int_w^k |s|^{p-2}\,ds$$
$$=(k-w)|s|^{p-2}s|_w^k$$
$$\leqslant \beta(p)(k-w)^2(|w|+|k|)^{p-2}$$

The evidence is now complete.

We define the following kind of mollification in time for the solution u's time regularity:

$$[u]_h(x,t) \overset{def}{=} \frac{1}{h}\int_0^t e^{\frac{s-t}{h}}u(x,s)ds \text{for any} u \in L^1(\Omega_T) \tag{7}$$

Lemma 2.3 (Properties of mollification) (Kinnunenet al., 2006**)**

1. If $u \in L^p(\Omega_T)$ then $\| [u]_h(x,t) \|_{L^p(\Omega_T)} \leqslant \| u \|_{L^p(\Omega_T)}$ and
$$\frac{\partial [u]_h}{\partial t} = \frac{u-[u]_h}{h} \in L^p(\Omega_T).$$
Moreover, $[u]_h \to u$ in $L^p(\Omega_T)$ as $h \to 0$.

2. If, in addition, $\nabla([u]_h) = [\nabla u]_h$ componentwise,
$$\| \nabla([u]_h) \|_{L^p(\Omega_T}\leqslant \| \nabla u \|_{L^p(\Omega_T)}$$
and $\nabla[u]_h \to \nabla u$ in $L^p(\Omega_T)$ as $h \to 0$.

3. Furthermore, if $u_k \to u$ in $L^p(\Omega_T)$ then also
$$[u_k]_h \to [u]_h and \frac{\partial [u_k]_h}{\partial t} \to \frac{\partial [u]_h}{\partial t}$$
in $L^p(\Omega_T)$. and $\nabla[u]_h \to \nabla u$ in $L^p(\Omega_T)$ as $h \to 0$.

4. If $\nabla u_k \to \nabla u$ in $L^p(\Omega_T)$ then also $\nabla[u_k]_h \to \nabla[u]_h \in L^p(\Omega_T)$.

5. Similar findings are true for weak convergence in $L^p(\Omega_T)$.

6. Finally, if $\varphi \in C(\overline{\Omega}_T)$ then
$$[\varphi]_h(x,t) + e^{-\frac{t}{h}}\varphi(x,0) \to \varphi(x,t)$$
uniformly in Ω_T as $h \to 0$.

In the sequel, we will use the following energy estimate (Trudinger, 1968).We briefly mention the estimate here before moving on to our major proof.

Proposition 2.4 *Consider that u is a locally weak subsolution to (1).Then a constant $\gamma(p)>0$ exists such that for all cylinders $Q_{R,S} = K_R(x_0) \times (t_0-S,t_0) \Subset \Omega_T$. There holds for every nonnegative, function for piecewise smooth cutting off piecewise smooth cut off function zeta disappearing on every $\partial K(x_0) \times (t_0-S,t_0)$, $k \in \mathbb{R}$*

$$ess\ sup_{t_0-S<t<t_0} \int_{K_R(x_0)\times\{t\}} \zeta^p A^{\pm}(k,u)\,dx + \iint_{Q_{R,S}} \zeta^p |D(u-k)_{\pm}|^p\,dx\,dt$$
$$\leqslant \gamma \iint_{Q_{R,S}} [|D\zeta|^p(u-k)_{\pm} + A^{\pm}(k,u)|\partial_t \zeta^p|]\,dx\,dt$$
$$+ \int_{K_R(x_0)\times\{t_0-S\}} \zeta^p A^{\pm}(k,u)\,dx \tag{8}$$

3 Positivity expansion

$\boldsymbol{K} \subset \mathbb{R}^{\boldsymbol{n}}$ and a cylinder $\boldsymbol{Q} \overset{\boldsymbol{def}}{=} \boldsymbol{K} \times (\boldsymbol{t_1},\boldsymbol{t_2}) \subset \boldsymbol{\Omega_T}$. We will use the following notations in this section such as

$$\mu^+ \geq ess\ sup_Q u, \mu^- \leq ess\ inf_Q u \quad \omega = \mu^+ - \mu^-$$

It is also assumed that $(x_0,t_0) \in Q$ for defining the forward cylinder

$K_{8\varrho}(x_0) \times (t_0, t_0 + (8\varrho)^p) \subset Q.$ (9)

Proposition 3.1 *Assume that u be a locally bounded, local weak sub(super) solution to (1) in Ω_T. Let there be some $(x_0, t_0) \in \Omega_T, M > 0, \alpha \in (0,1)$ and $\varrho > 0$. If*

$$|\{\pm(\mu^{\pm} - u(.,t_0)) \geq M\} \cap K_\varrho(x_0)| \geq \alpha|K_\varrho|.$$

and the hypotheses (9) are true. Then there are constants ξ, δ and $\eta \in (0,1)$ that depend exclusively on the data and α, such that either

$$|\mu^{\pm}| > \xi M$$

or

$$\pm(\mu^{\pm} - u) \geq \eta M \ a.e. in \ K_{2\varrho}(x_0) \times (t_0 + \delta(\frac{1}{2}\varrho)^p, t_0 + \delta\varrho^p),$$

where

$$\xi = \begin{cases} 2\eta, & if \ p > 2, \\ 8, & if \ 1 < p \leq 2. \end{cases}$$

The three lemmas that are presented in the following sections provide as direct evidence for the Proposition 3.1.

3.1 Extension of Positivity in Measure

Lemma 3.2 *If $M > 0$ and $\alpha \in (0,1)$. Afterwards, there are δ and ε in $(0,1)$, that solely depend on the data and α, in which case whenever u is a locally bounded local weak sub(super)-solution to (1) in Ω_T satisfying*

$$|\{\pm(\mu^{\pm} - u(\cdot,t_0)) \geq M\} \cap K_\varrho(x_0)| \geq \alpha|K_\varrho|$$

, then either

$$|\mu^{\pm}| > 8M$$

or

$$|\{\pm(\mu^{\pm} - u(\cdot,t)) \geq \varepsilon M\} \cap K_\varrho(x_0)| \geq \frac{\alpha}{2}|K_\varrho| \ forall \ t \in (t_0, t_0 + \delta\varrho^p) \quad (10)$$

Proof. Here, the context of super solutions will be demonstrated, and the other instance of sub solutions can be handled similarly. At first consider $(x_0, t_0) = (0,0)$ and $|\mu^-| \leq 8M$. Otherwise there is nothing to prove. We will apply the energy inequality (2.4) in the cylinder $Q = K_\varrho \times (0, \delta\varrho^p]$, with $k = \mu^- + M$ and choose the nonnegative time independent cutoff function $\zeta(x,t) \equiv \zeta(x)$ which equals 1 on $K_{(1-\sigma)\varrho}$ with $\sigma \in (0,1)$ and vanishes on the the boundary ∂K_ϱ satisfying $|D\zeta| \leq (\sigma\varrho)^{-1}$. Then for all $0 < t < \delta\varrho^p$, we have $\int_{K_{\varrho\times\{t\}}} \int_u^k |s|^{p-2}\zeta^p(s-k)_- \, ds \, dx$

$$\leq \int_{K_{\varrho\times\{0\}}} \int_u^k |s|^{p-2}\zeta^p(s-k)_- \, ds \, dx + \gamma \iint_Q |D\zeta|^p(u-k)_-^p \, dx \, dt. \quad (11)$$

We will estimate the above integral separately, for the first integral, at the initial time level, we'll use the measure theoretical information. and applying $u \geq \mu^-$ yield.

$$\int_{K_{\varrho\times\{0\}}} \int_u^k |s|^{p-2}\zeta^p(s-k)_- \, ds \, dx \leq (1-\alpha)|K_\varrho| \int_{\mu^-}^k |s|^{p-2}(s-k)_- \, ds.$$

The second term on the right of (11) is approximated by

$$\iint_Q |D\zeta|^p(u-k)_-^p \, dx \, dt \leq \frac{M^p}{(\sigma\varrho)^p}|Q| = \frac{\delta M^p}{\sigma^p}|K_\varrho|$$

The left part of (11) can be estimated from below by

$$\int_{K_{\varrho\times\{t\}}} \int_u^k |s|^{p-2}\zeta^p(s-k)_- \, ds \, dx \geq |A_{k_\varepsilon,(1-\sigma)\varrho}(t)| \int_{k_\varepsilon}^k |s|^{p-2}(s-k)_- \, ds$$

where

$$A_{k_\varepsilon,(1-\sigma)\varrho}(t) = \{u(\cdot,t) \leq k_\varepsilon\} \cap K_{(1-\sigma)\varrho}, and \ k_\varepsilon = \mu^- + \varepsilon M$$

with $\varepsilon \in (0, \frac{1}{2})$ to be established later. For the reason that Lemma 2.2 and $\frac{1}{2}M \leq (1-\varepsilon)M = k - k_\varepsilon \leq |k_\varepsilon| + |k| \leq 2(|\mu^-| + M) \leq 18M$ with further computation from below

$$\int_{k_\varepsilon}^k |s|^{p-2}(s-k)_- \, ds = \frac{1}{p-1}A^-(k, k_\varepsilon) \geq \frac{1}{\gamma(p)}(|k_\varepsilon| + |k|)^{p-2}(k-k_\varepsilon)^2 \geq \frac{1}{\gamma(p)}M^p.$$

(12) We should mention here that

$$A_{k_\varepsilon,\varrho}(t) = |A_{k_\varepsilon,(1-\sigma)\varrho}(t)| \cup (A_{k_\varepsilon,\varrho}(t) \backslash A_{k_\varepsilon,(1-\sigma)\varrho}(t))|$$
$$\leq |A_{k_\varepsilon,(1-\sigma)\varrho}(t)| + |K_\varrho \backslash K_{(1-\sigma)\varrho}|$$
$$\leq |A_{k_\varepsilon,(1-\sigma)\varrho}(t)| + N\sigma|K_\varrho|$$

Aggregating all the above estimates yields that

$$|A_{k_\varepsilon,\varrho}(t)| \le \frac{\int_{\mu^-}^{k} |s|^{p-2}(s-k)_- \, ds}{\int_{k_\varepsilon}^{k} |s|^{p-2}(s-k)_- \, ds}(1-\alpha)|K_\varrho| + \frac{\gamma\delta}{\sigma^p}|K_\varrho| + N\sigma|K_\varrho|$$

for a constant $\gamma = \gamma(p)$. The above inequality's fractional number can be worded as follows

$$1 + I_\varepsilon \quad where \quad I_\varepsilon = \frac{\int_{\mu^-}^{k_\varepsilon} |s|^{p-2}(s-k)_- \, ds}{\int_{k_\varepsilon}^{k} |s|^{p-2}(s-k)_- \, ds}$$

While observing $\mu^-| \le 8M$ and $|k_\varepsilon| \le 9M$ and Lemma 2.1, implies

$$\int_{\mu^-}^{k_\varepsilon} |\tau|^{p-2}(\tau-k)_- \, d\tau \le M \int_{\mu^-}^{k_\varepsilon} |\tau|^{p-2} \, d\tau = M|s|^{p-2}s|_{\mu^-}^{k_\varepsilon} \le \gamma(p)M^p\varepsilon$$

Together with inequality (12) we obtain

$$I_\varepsilon \le \gamma(p)\varepsilon.$$

This enables us to quantitatively select the various parameters. Infact, we may select a $\varepsilon \in (0,1)$ small enough to

$$(1-\alpha)(1+\gamma\varepsilon) \le 1 - \frac{3}{4}\alpha$$

we then define $\sigma := \frac{\alpha}{8N}$. Last but not least, we select a $\delta \in (0,1)$ sufficient in magnitude that

$$\frac{\gamma\delta}{\sigma^p} \le \frac{\alpha}{8}.$$

Notably, this specifies δ as a constant that depends on the data and and α .With these options, $|A_{k_{\varepsilon,\sigma}}(t)| \le (1-\frac{\alpha}{2})|K_\varrho|$. This demonstrates the purported spread of positivity (10), so long as $0 < t < \delta\varrho^p$.

$$\int_{\mu^-}^{k_\varepsilon} |\tau|^{p-2}(\tau-k)_- \, d\tau \le M \int_{\mu^-}^{k_\varepsilon} |\tau|^{p-2} \, d\tau = M|s|^{p-2}s|_{\mu^-}^{k_\varepsilon} \le \gamma(p)M^p\varepsilon$$

Together with inequality (12) we obtain

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Notably, this specifies δ as a constant that depends on the data and α.With these options, $|A_{k_{\varepsilon,\sigma}}(t)| \le (1-\frac{\alpha}{2})|K_\varrho|$. This demonstrates the purported spread of positivity (10), so long as $0 < t < \delta\varrho^p$.

3.2 Lemma of shrinking

Lemma 3.3 *Consider that the second alternative (10) in Lemma 3.2 is true, let* $Q = K_\varepsilon(x_0) \times (t_0, t_0 + \delta\varepsilon^p]$ *be the corresponding cylinder and let* $\tilde{Q} = K_{4\varepsilon}(x_0) \times (t_0, t_0 + \delta\varrho^p] \subset \Omega_T$. *For any positive integer* j_* *there is a* $\gamma > 0$ *that solely relies on the data and* α, *if* $1 < p < 2$, *we have*

$$\left|\left\{\pm(\mu^\pm - u) \le \frac{\varepsilon M}{2^{j_*}}\right\} \cap \tilde{Q}\right| \le \frac{\gamma^{\frac{p-1}{p}}}{j_*}|\tilde{Q}|,$$

However, if $p > 2$, the same conclusion is valid as long as $|\mu^\pm| < \varepsilon M 2^{-j_*}$.

Proof. Since the scenario with subsolutions is quite resemblance, we only present the example of super-solutions. Additionally, we consider $(x_0, t_0) = (0,0)$. The energy approximation is used in Proposition 2.4 in $K_{8\varrho} \times (0, \delta\varrho^p)$ having levels

$$k_j := \mu^- + \frac{\varepsilon M}{2^j}, \quad j = 0,1,2, \cdots j_*,$$

and initiate a cutoff function ζ in $K_{8\varrho}$ that equals 1 in $K_{4\varrho}$ and disappears on $\partial K_{8\varrho}$, in such a way $|D\zeta| \le \varrho^-$. then we attain

$$\iint_{\tilde{Q}} |D(u-k_j)_-|^p \, dx\, dt \le \int_{K_{8\varrho}\times\{0\}} A^-(k_j, u)\, dx + \frac{\gamma}{\varrho^p}\iint_{K_{8\varrho}\times(0,\delta\rho^p]} (u - k_j)_-^p \, dx\, dt$$

We now handle each term on the right side independently. The first one is where we start. Lemma 2.2 provides us with

$$A^-(k_j, u) \le \gamma(|u| + |k_j|)^{p-2}(u-k_j)_-^2$$

When $P \ge 2$, we use $(u-k_j)_- \le |u| + |k_j|$ as well as $u \ge \mu^-$ and $|\mu^-| \le \varepsilon M 2^{-j_*}$ to estimate

$$A^-(k_j, u) \le \gamma(|u| + |k_j|)^p \chi_{u \le k_j} \le \gamma\left(\frac{\varepsilon M}{2^j}\right)^p$$

when $1 < p < 2$, we again use $(u - k_j)_- \le |u| + |k_j|$ and $u > \mu^-$ to obtain

$$A^-(k_j, u) \le \gamma(u - k_j)_-^p \le \gamma\left(\frac{\varepsilon M}{2^j}\right)^p,$$

for a constant γ that is solely dependent on p. This indicates, specifically, that

$$A^-(k_j, u) \le \frac{\gamma}{\delta\varrho^p}\left(\frac{\varepsilon M}{2^j}\right)^p |\tilde{Q}|$$

in every instance, it is true. We use the bound to calculate the second integral on the right side of the energy estimate $(u - k_j)_- \le \varepsilon M 2^{-j}$. As a result, the above calculation always produces

$$\iint_{\tilde{Q}} |D(u - k_j)_-|^p \, dx\, dt \le \frac{\gamma}{\delta\varrho^p}\left(\frac{\varepsilon M}{2^j}\right)^p |\tilde{Q}|$$

We then use [(DiBenedetto, 1993), Chapter I, Lemma 2.2] slice based on $u(\cdot, t)$ for $t \in (0, \delta\varrho^p]$ over the cube K_ϱ, the levels $k_{j+1} < k_j$. Considering the measure theoretical information

$$|\{u(\cdot,t) > \mu^- + \varepsilon M\} \cap K_\varrho| \ge \frac{\alpha}{2}|K_\varrho| \text{ for}$$

all $t \in (0, \delta\varrho^p]$, this gives

$$\begin{aligned}
&(k_j - k_{j+1})|\{u(\cdot,t) < k_{j+1}\} \cap K_{4\varrho}| \\
&\le \frac{\gamma\varrho^{n+1}}{|\{u(\cdot,t) > k_j\} \cap K_{4\varrho}|}\int_{\{k_{j+1} < u(\cdot,t) < k_j\} \cap K_{4\varrho}} |Du(\cdot,t)| \, dx \\
&\le \frac{\gamma\varrho}{\alpha}\left[\int_{\{k_{j+1} < u(\cdot,t) < k_j\} \cap K_{4\varrho}} |Du(\cdot,t)|^p \, dx\right]^{\frac{1}{p}} |\{k_{j+1} < u(\cdot,t) < k_j\} \cap K_{4\varrho}|^{1-\frac{1}{p}} \\
&= \frac{\gamma\varrho}{\alpha}\left[\int_{\{k_{j+1} < u(\cdot,t) < k_j\} \cap K_{4\varrho}} |D(u - k_j)_-(\cdot,t)|^p \, dx\right]^{\frac{1}{p}} [|A_j(t)| - |A_{j+1}(t)|]^{1-\frac{1}{p}}.
\end{aligned}$$

Here, the short hand notation was utilized in the last line. $A_j(t) := \{u(\cdot,t) < k_j\} \cap K_{4\varrho}.$ We now incorporate the last inequality regarding to t over $(0, \delta\varrho^p]$ and utilize Hölder's inequality over time. Using the acronym $A_j = \{u < k_j\} \cap \tilde{Q}$ results in

$$\begin{aligned}
\frac{\varepsilon M}{2^{j+1}}|A_{j+1}| &\le \frac{\gamma\varrho}{\alpha}\left[\iint_{\tilde{Q}} |D(u - k_j)_-(\cdot,t)|^p \, dx\right]^{\frac{1}{p}} [|A_j(t)| - |A_{j+1}(t)|]^{1-\frac{1}{p}} \\
&\le \gamma\left(\frac{\varepsilon M}{2^j}\right)|\tilde{Q}|^{\frac{1}{p}}[|A_j(t)| - |A_{j+1}(t)|]^{1-\frac{1}{p}}
\end{aligned}$$

Keep in mind that δ is dependent on the data and α. γ thus only depends on the data and α. Raise the power $\frac{p}{p-1}$ on both sides of the aforementioned inequality to obtain

$$|A_{j+1}|^{\frac{p}{p-1}} \le \gamma|\tilde{Q}|^{\frac{1}{p-1}}[|A_j| - |A_{j+1}|].$$

From 0 to $j_* - 1$, Together, these inequalities, provides as

$$j_*|A_{j_*}|^{\frac{p}{p-1}} \le \gamma|\tilde{Q}|^{\frac{p}{p-1}}.$$

As a result, we conclude

$$|A_{j_*}| \le \frac{\gamma}{j_*^{\frac{p-1}{p}}}|\tilde{Q}|.$$

The proof is now complete.

3.3 Lemma of the DeGiorgi type

On cylinders of the form $\boldsymbol{Q_\varrho(\theta)}$, we demonstrate a lemma of the DeGiorgi type here. $\boldsymbol{\theta}$ will be a constant used in the application that is independent of the solution and solely depends on the data.

Lemma 3.4 *If* $\boldsymbol{u}$ *is a locally bounded, local sub(super)-solution to* $(\mathbf{1})$ *in* $\boldsymbol{\Omega_T}$, *and* $(\boldsymbol{x_0}, \boldsymbol{t_0}) + \boldsymbol{Q_\varrho(\theta)} = \boldsymbol{K_\varrho(x_0)} \times (\boldsymbol{t_0} - \boldsymbol{\theta\varrho^p}, \boldsymbol{t_0}] \Subset \boldsymbol{\Omega_T}$. *There is a constant called* $\boldsymbol{\nu} \in (\mathbf{0}, \mathbf{1})$, *which solely depends on the data and* $\boldsymbol{\theta}$, *such that if*

$$|\{\pm(\mu^\pm - u) \le M\} \cap (x_0, t_0) + Q_\varrho(\theta)| \le \nu|Q_\varrho(\theta)|.$$

then either

$$|\mu^\pm| > 8M,$$

or

$$\pm(\mu^\pm - u) \ge \frac{1}{2}M \quad a.e. in \quad (x_0, t_0) + Q_{\frac{1}{2}\varrho}(\theta).$$

Proof. Solely the situation involving super-solutions is established; the situation involving sub-solutions is analogous. Assume that

$(x_0, t_0) = (0,0)$ and $|\mu^- \le 8M|$. If not, nothing to be established. We first note that due to Lemma 2.2, we have to use the energy estimate in Proposition 2.4, we get

$$A^-(k,u) \le \gamma(|u| + |k|)^{p-2}(u-k)_-^2 \le \gamma(|u| + |k|)^{p-1}(u-k)_-$$

and for $\tilde{k} < k$ there holds $(u-k)_- \ge (u-\tilde{k})_-$. Therefore, the energy estimate yields

$$ess\ sup_{-\theta\varrho^p<t<0}\int_{K_\varrho} \zeta^p(|u| + |k|)^{p-2}(u-\tilde{k})_-^2\ dx + \iint_{Q_\varrho(\theta)} \zeta^p|D(u-\tilde{k})_-|^p\,dx\,dt$$

$$\le \gamma\iint_{Q_\varrho(\theta)} |D\zeta|^p(u-k)_-^p\ dx\,dt + \gamma\iint_{Q_\varrho(\theta)} (|u| + |k|)^{p-1}(u-k)_-|\partial_t\zeta^p|\,dx\,dt$$

for any piecewise smooth cutoff function ζ that is not negative and vanishes on the parabolic edge of $Q_\varrho(\theta)$ To make use of this energy estimation, we establish

$$\begin{cases} k_n = \mu^- + \frac{M}{2} + \frac{M}{2^{n+1}}, & \tilde{k}_n = \frac{k_n + k_{n+1}}{2}, \\ \varrho_n = \frac{\varrho}{2} + \frac{\varrho}{2^{n+1}}, & \tilde{\varrho}_n = \frac{\varrho_n + \varrho_{n+1}}{2} \\ K_n = K_{\varrho_n}, & \tilde{K}_n = K_{\tilde{\varrho}_n}(\theta) \\ Q_n = Q_{\varrho_n}(\theta) & \bar{Q}_n = Q_{\tilde{\varrho}_n}(\theta) \end{cases} \tag{13}$$

Recall that $Q_{\varrho_n}(\theta) = K_n \times (-\theta\varrho_n^p, 0]$ and $Q_{\tilde{\varrho}_n}(\theta) = \tilde{K}_n \times (-\theta\tilde{\varrho}_n^p, 0]$. Initiate the cutoff function $0 \le \zeta \le 1$ disappearing at the parabolic edge of Q_n and equal to identity in $\bar{Q}_n$, such that

$$|D\zeta| \le \gamma\frac{2^n}{\varrho}\ \ and\ \ |\zeta_t| \le \gamma\frac{2^{pn}}{\theta\varrho^p}.$$

The energy estimations in this scenario may be expressed as

$$ess\ sup_{-\theta\tilde{\varrho}_n^p<t<0}\int_{\tilde{K}_n} (|u| + |k_n|)^{p-2}(u-\tilde{k}_n)_-^2\ dx + \iint_{\bar{Q}_n} |D(u-\tilde{k}_n)_-|^p\ dx\,dt$$

$$\le \gamma\frac{2^{pn}}{\varrho^p}\iint_{Q_n} (u-k_n)_-^p\ dx\,dt + \gamma\frac{2^{pn}}{\theta\varrho^p}\iint_{Q_n} (|u| + |k_n|)^{p-1}(u-k_n)_-\ dx\,dt$$

$$\le \gamma\frac{2^{pn}}{\varrho^p}M^p|A_n|,$$

where γ based on the data and θ. We made use of $\mu^- \le u \le k_n \le \mu^- + M$ on $|A_n|$, where

$$A_n = \{u < k_n\} \cap Q_n.$$

However, we do remember $|\mu^-| \le 8M$, in order that $u \le \tilde{k}_n$ implies $|u| + |k_n| \le 18M$ and $|u| + |k_n| \ge k_n - u \ge k_n - \tilde{k}_n = 2^{-(n+3)}M$. Inserting this above, we find that

$$\frac{M^{p-2}}{2^{p(n+3)}}ess\ sup_{-\theta\tilde{\varrho}_n^p<t<0}\int_{\tilde{K}_n} (u-\tilde{k}_n)_-^2\ dx + \iint_{\bar{Q}_n} |D(u-\tilde{k}_n)_-|^p\ dx\,dt \le \gamma\frac{2^{pn}}{\varrho^p}M^p|A_n|, \tag{14}$$

Making use of Hölder inequality and the Sobolev imbedding [5, Chapter I, Proposition 3.1] to analyze the cutoff function of $0 \le \phi \le 1$,which disappears at the parabolic edge of $\bar{Q}_n$ and equals the identity in Q_{n+1}, results in that

$$\frac{M}{2^{n+3}|A_{n+1}|} \le \iint_{\bar{Q}_n} |(u-\tilde{k}_n)_-|\phi\ dx\,dt$$

$$\le \left[\iint_{\bar{Q}_n} [u-\tilde{k}_{n\_}\phi]^{p\frac{N+2}{N}}\ dx\,dt\right]^{\frac{N}{p(N+2)}} |A_n|^{1-\frac{N}{p(N+2)}}$$

$$\le \gamma\left[\iint_{\bar{Q}_n} |D[u-\tilde{k}_{n\_}\phi]|^p\ dx\,dt\right]^{\frac{N}{p(N+2)}} \times \left[ess\ sup_{-\theta\tilde{\varrho}_n^p<t<0}\int_{\tilde{K}_n} (u-\tilde{k}_n)_-^2\ dx\right]^{\frac{1}{N+2}} |A_n|^{1-\frac{N}{p(N+2)}}$$

$$\le \gamma\left(\frac{2^{pn}M^p}{\varrho^p}\right)^{\frac{N}{p(N+2)}}\left(\frac{2^{p(2n+3)}M^2}{\varrho^p}\right)^{\frac{1}{N+2}} |A_n|^{1+\frac{1}{(N+2)}}$$

$$= \gamma\frac{2^{\frac{(2p+N)n}{N+2}}}{\varrho^{\frac{N+p}{N+2}}}M|A_n|^{1+\frac{1}{(N+2)}}.$$

The estimated energy was utilised in the last line.

With regard to $Y_n = \frac{|A_n|}{|Q_n|}$, One alternative for this is

$$Y_{n+1} \leq \gamma b^n Y_n^{1+\frac{1}{N+2}},$$

for a fixed γ that is solely dependent on the data and with $b \equiv 2^{\frac{2(p+N+1)n}{N+2}}$. Hence, by [(DiBenedetto, 1993), Chapter I, Lemma 4.1], a positive constant ν exists, purely based on data, this way $Y_n \to 0$ if we insist on that $Y_0 \leq \nu$, that is equivalent to assuming

$$|A_0| = |\{u < k_0\} \cap Q_0| = |\{u < \mu^- + M\} \cap Q_\varrho(\theta)| \leq \nu |Q_\varrho(\theta)|.$$

As a result of $Y_n \to 0$ in the limit $n \to \infty$ we now have

$$|\{u \leq \mu^- + \frac{1}{2}M\} \cap Q_{\frac{1}{2}\varrho}(\theta)| = 0.$$

The lemma's proof is finished at this step.

3.4 Proof of Our Result

We have all the necessary materials on hand right now to demonstrate the growth of positivity.

Proof. Only the case of super-solutions is presented here; the case of sub-solutions is identical. Consider $(x_0, t_0) = (0,0)$. Depending on the data, we express the respective constants from Lemma 3.2 and 3.3 by $\delta, \varepsilon \in (0,1)$ and $\gamma > 0$, whereas α and $\nu \in (0,1)$ indicate the constant from the application of Lemma 3.4 with $\theta = \delta$.Then, we choose an integer j_* in a way that

$$\frac{\gamma}{j_*^{\frac{p-1}{p}}} \leq \nu.$$

We let $\zeta = 8$ if $1 < p < 2$ and $\xi = \varepsilon 2^{-j_*}$ if $p > 2$. Since there is nothing to establish otherwise, we can assume in the following that $|\mu^-| \leq \xi M$. Applying Lemma 3.2 and Lemma 3.3 in sequence, we arrive to the conclusion that

$$|\{u \leq \mu^- + \frac{\varepsilon M}{2^{j_*}}\} \cap \tilde{Q}| \leq \nu |\tilde{Q}|,$$

where $\tilde{Q} = K_{4\varrho} \times (0, \delta\varrho^p]$. Applying Lemma 3.4 with M in place of $\frac{\varepsilon M}{2^{j_*}}$ provides

$$u \geq \mu^- + \frac{\varepsilon M}{2^{j_*}+1} \;\; a.e. in \;\; K_{2\varrho} \times (\delta(\frac{1}{2}\varrho)^p, \delta\varrho^p].$$

This demonstrates what Proposition 3.1 claims for $\eta = \frac{\varepsilon}{2^{j_*+1}}$. Here we should make clear that we have made our choice $\xi = 2\eta$ with $p > 2$.

Thus we have proved our findings as mention one more time below

Assume that u is a locally bounded, local weak sub(super) solution to the equation (1) within the domain Ω_T. Here, $(x_0, t_0) \in \Omega_T$, $M > 0$, $\alpha \in (0,1)$, and $\varrho > 0$. If the conditions (9) and the following inequality are satisfied:

$$|\{\pm(\mu^\pm - u(., t_0)) \geq M\} \cap K_\varrho(x_0)| \geq \alpha |K_\varrho|.$$

Then, there exist constants ξ, δ, and η, all within the range of (0, 1), which depend solely on the given data and α. These constants have the property that either:

$$|\mu^\pm| > \xi M$$

or

$$\pm(\mu^\pm - u) \geq \eta M \;\; a.e. in K_{2\varrho}(x_0) \times (t_0 + \delta(\frac{1}{2}\varrho)^p, t_0 + \delta\varrho^p),$$

In the degenerate case, ξ is equal to 2η, and in the singular case, ξ equals 8

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