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Analysis of Local Bifurcations in One-Dimensional Systems

Research Article

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ABSTRACT

This article delves deeper into the study of local bifurcation in maps, although only in the context of a single dimension. The various local bifurcation types common to one-dimensional maps each have their own specific conditions that must be fulfilled. An object divides into two when it bifurcates. After a bifurcation, a family of one-parameter functions retains its stable or cyclic point structure. A bifurcation in the iterative process happens when a parameter is altered, causing a qualitative change in behavior. The phenomena of local splitting can be induced by adjusting just one parameter. Both the norm form of the transcritical split and the stability of the saddle node are highlighted. As with supercritical and subcritical bifurcations, the stability of the pitchfork bifurcation is assessed, as is its norm form.

Keywords: *Local bifurcation, Bifurcation point, Stability, Saddle-node bifurcation, Transcritical bifurcation, Pitchfork bifurcation*

1. Introduction

The distinctive configuration of a dynamical system not only offers specialized analytical methods tailored to that system, but also significantly constrains the range of potential dynamics. The significance of these atypical structures is heightened by their increasing frequency of occurrence. One-dimensional autonomous map bifurcation theory is well-known and found in many discrete dynamical system studies (Cushing et al., 2003; Elaydi, 2008; Iooss, 1979; Robinson, 1999; Wiggins, 2003; Kuznetsov, 2004; Crawford, 1991; Hale &Kocak, 1991). Differential equations have been used to assess a wide variety of demographic models. The capabilities of discrete-time platforms may be useful in computational simulations (Ahmad, 1993; Tang & Zou, 2006). If the population size is steady during the lifespan or remains small over a number of generations, discrete-time models tend to most accurately reflect the fluctuations in population dynamics (Zhou & Zho, 2003; Liu, 2010). Using difference equations

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to model interactions across species with non overlapping life cycles makes such models more amenable to analysis of action patterns (Freedman, 1980; Agarwal, 2000; Goh, 1980). When compared to their continuous-time equivalents, numerical simulations of discrete-time models offer greater speed of computation and a greater number of fluctuating behavior representations (Zhao et al., 2011; Flores, 2011; Hone et al., 2010). There is no doubt that discrete-time models have significant advantages over continuous-time ones in situations in which the generations that make up a population never overlap. Chaotic behavior, cyclical windows, and period-doubling bifurcations are all characteristics seen in the Ricker curve (Azizi & Kerr, 2020; Ricker, 1954; Azizi & Kerr, 2020; Azizi, 2015). There exist trajectories that span a sizeable piece of the irreducible circle and periodic orbits that are unaffected by one another. A closed invariant curve is produced by both super-critical and sub-critical bifurcations, the former generating an unstable curve and the latter yielding a curve that are inflexible (Lei, 2018). Different forms of comparable species display a wide variety of behaviors and physical characteristics(Chen et al., 2012). Ecological modeling at several time scales is thus more useful. Over the preceding decade, researchers have examined the phase's hierarchy model of species evolution in great detail (Li & Chen, 2017; Xiao et al., 2019). Bifurcation analysis is often necessary to completely understand the parameter dynamics of ODE models used in mathematical biology. In recent decades, "mechanobiology" has come to encompass the study of the physical features and activities of cells in addition to their biological counterparts. Biological models of cellular behavior have been developed as a result of this curiosity and are being put to use in biochemical and mechanical studies(Rajagopal et al., 2007). This alleviated their worry, but more work is needed to fully understand the mathematical model's behavior spanning a wide variety of inputs. Given the rarity of homoclinic solutions and saddle-node bifurcations in co dimension two, we were curious to see if Zmurchok et al.'s model might produce these phenomena (Zmurchok et al., 2018). The images here show how transcritical and pitchfork bifurcations frequently lead to aberrations that break symmetry. Analytic bifurcation concept is used to precisely specify the form of the disrupted local bifurcation schematics (Crandall & Rabinowitz, 1971; Crandall & Rabinowitz, 1973). A number of studies (Kirchg & Sorger, 1969; Nirenberg, 1981; Ruelle, 1973; Sattinger, 1971) that employs topological techniques to address bifurcation issues are referenced. It is investigated the circumstances under which the bifurcation of a unique simple equilibrium produces a vast family of simple non-hyperbolic equilibrium with non-zero indices. The pitchfork bifurcation, proposed by (Crandall & Rabinowitz, 1971; Chow & Hale, 1982) and described by Crandall & Rabinowitz (1973), assumes that each parameter has no less than one zero near the bifurcation point. As Crandall and Rabinowitz devised pitchfork splitting. A discrete dynamical system that has the potential to summaries these data according to yet another criterion and offer an environment in which the critical curve may be discovered with relative ease. Because of the existence of reflecting symmetry, such a condition in an ongoing dynamical system is referred to as a pitchfork bifurcation (Monroe, 1992). This is because the reflection symmetry is present. The very same criterion can also be used to provide a one-of-a-kind illustration of Monroe's hypothesis by Ganikhodjaev et al. (2003).

2. Saddle-node bifurcation

A saddle-node is a dynamical system with two concealed contending nodes. The trace inversion makes the bifurcation a saddle-node bifurcation. A saddle-node bifurcation may develop at the pivot's centre. The two equivalence elements in an a single-dimensional cycle rest are saddle and node. Bifurcation happens when one scheme parameter causes another to change fixed point stability or permanence. A suspicious object that remains cohesive on a vertical beam's turret is a good naive example of a dynamical system bifurcation. The item's mass may determine it. As the object's mass grows, the beam angle, x, remains roughly constant. The beam's mass approaching the bifurcation point will have a severe impact. Adjusting one control parameter changed the system's behavior. The saddle-node bifurcation (Layek, 2015). can be observed in the system $\dot{x}(t) = f(x,\rho),\ x,\rho \in \Re$ (1)

at (x,ρ) if the following conditions are met at the equilibrium point x at ρ.

$$f(0,0) = 0 \tag{2}$$

$$\left[\frac{\partial f}{\partial x}\right]_{\rho=0,x=0} = 0 \tag{3}$$

$$\left[\frac{\partial f}{\partial \rho}\right]_{\rho=0,x=0} \neq 0 \tag{4}$$

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{\rho=0,x=0} \neq 0 \tag{5}$$

Consider the one-dimensional equation of the form

$$f(x,\rho) = \frac{dx}{dt} = \rho + x^2;\ x \in \Re \tag{6}$$

Put an ρ in front of it to parameterize it. The steady-state values for (6) are found by solving for

$$f(x,\rho) = 0$$

$$\Rightarrow \rho + x^2 = 0$$

$$\Rightarrow x^2 = -\rho$$

$$\therefore x^*_{1,2} = \pm\sqrt{-\rho} \tag{7}$$

There are three possible results; depending on how we set the parameter ρ .When $\rho < 0$, the system has two fixed points. They add up to one another at $x^* = 0$ when $\rho = 0$, but disappear when $\rho > 0$. We will factor in underflow in the real line. The velocity vector $\dot{x}$ at any point x in the flow can be found by solving the system $\dot{x} = f(x,\rho)$, where $f(x,\rho)$ is the real-line representation of the vector field describing the flow.

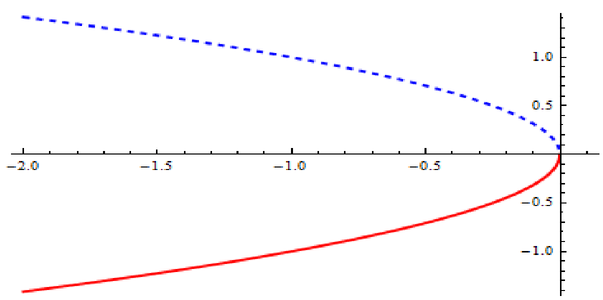


Fig. 1 Bifurcation Diagram for a Saddle Node Bifurcation of the equation (7).

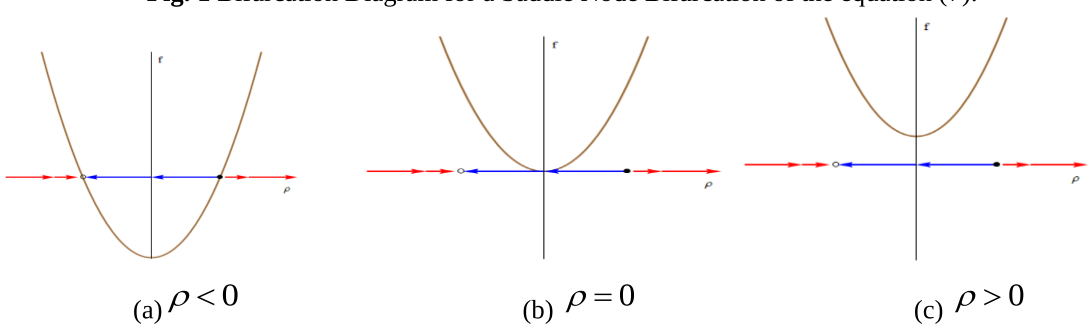


Fig. 2 Flow diagrams for the saddle node bifurcation of the equation (6) and (a) $\rho < 0$, (b) $\rho = 0$, and (c) $\rho > 0$.

The arrow will point to the left if $f > 0$, and to the right if $f < 0$. In this case, the flow is to the right because f is greater than zero, and it is to the left because f is less than zero. Flow is unaffected by a change in $f=0$. We call the locations of the system's fixed points or equilibrium points (6) where $f=0$ because there is no flow at those locations. These are the locations where the system is stable, or in equilibrium. Figure (2) depicts the parabola-like shape that the graph of the vector field $f(x,\rho)$ takes in the $\rho-f$ plane. Figure 2(a) depicts two fixed points of the systems for the case where $\rho<0$. According to the flow imagination, the figure should show an unstable fixed point at $x=\sqrt{-\rho}$ and a stable fixed point at $x=-\sqrt{-\rho}$. Parabola rises and the two fixed points move closer together until they meet at $x=0$ and $\rho=0$, as can be seen in the figure. For $\rho>0$, the system does not settle into any unique configuration, as depicted in figure 2(c). The dynamics of this very basic system are utterly mesmerizing, despite their apparent simplicity. Since the vector fields for $\rho<0$ and $\rho>0$ are qualitatively different from one another, the dynamics underwent a bifurcation when $\rho=0$. The bifurcation point, or turning point of the trajectory, is the location along the trajectory at which the parameter ρ has a value of zero. The bifurcation diagram of the system depicts the connection between ρ and x^*, the fixed point of the system. The term "saddle-node bifurcation" is used to describe this specific type of bifurcation.

2.1 (a) Regarding the equation (6), the saddle-node bifurcation can be written in its normal form.

In the event that we presume the system (1) has a point of equilibrium at $x=x_0$ for $\rho=\rho_0$ in a region where the saddle-node bifurcation requirements are satisfied from the equation (2), (3), (4), and (5), then this would suggest that the point of equilibrium is situated at a saddle-node. The result that we get when we extend the function $f(x,\rho)$ in a Taylor series in the area surrounding the point (x_0,ρ_0) is as follows:

$$\dot{x}=f(x,\rho)$$

$$=f(x_0,\rho_0)+(x-x_0)\left[\frac{\partial f}{\partial x}\right]_{(x_0,\rho_0)}+(\rho-\rho_0)\left[\frac{\partial f}{\partial \rho}\right]_{(x_0,\rho_0)}+\frac{1}{2!}(x-x_0)^2\left[\frac{\partial^2 f}{\partial x^2}\right]_{(x_0,\rho_0)}$$

$$+(x-x_0)(\rho-\rho_0)\left[\frac{\partial^2 f}{\partial x\partial \rho}\right]_{(x_0,\rho_0)}+\frac{1}{2!}(\rho-\rho_0)^2\left[\frac{\partial^2 f}{\partial \rho^2}\right]_{(x_0,\rho_0)}+\ldots$$

Using the conditions of (2), (3), (4), and (5) the above equation contains

$$\dot{x}=f(x,\rho)$$

$$=(\rho-\rho_0)\left[\frac{\partial f}{\partial \rho}\right]_{(x_0,\rho_0)}+\frac{1}{2!}(x-x_0)^2\left[\frac{\partial^2 f}{\partial x^2}\right]_{(x_0,\rho_0)}+\ldots \tag{8}$$

From the system (6)

$$f(x,\rho)=\frac{dx}{dt}=\rho+x^2;\ x\in\Re \tag{9}$$

Differentiating (9) with respect to ρ

$$\left[\frac{\partial f}{\partial \rho}\right]_{(x_0,\rho_0)}=1$$ [Taking $x_0=0$ and $\rho_0=0$]

$$\left[\frac{\partial f}{\partial \rho}\right]_{(0,0)}=1$$

Differentiating (9) with respect to x

$$\left[\frac{\partial f}{\partial x}\right]_{(x_0,\rho_0)}=2x \tag{10}$$

[Taking $x_0=0$ and $\rho_0=0$]

$$\left[\frac{\partial f}{\partial x}\right]_{(0,0)}=0$$

Differentiating (10) with respect to x

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{(x_0,\rho_0)}=2$$

[Taking $x_0=0$ and $\rho_0=0$]

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{(0,0)}=2$$

Putting the above values from (8) we get,

$$\dot{x}=f(x,\rho)$$

$$=(\rho-0)\left[\frac{\partial f}{\partial \rho}\right]_{(0,0)}+\frac{1}{2!}(x-0)^2\left[\frac{\partial^2 f}{\partial x^2}\right]_{(0,0)}+\ldots$$

$$=\rho.1+\frac{1}{2!}x^2\times 2+\ldots$$

$$=\rho.\phi+\frac{1}{2!}x^2\times\varphi+\ldots \qquad (11)$$

In this equation, both $\phi=\left[\frac{\partial f}{\partial \rho}\right]_{(0,0)}=1$ and $\varphi=\left[\frac{\partial^2 f}{\partial x^2}\right]_{(0,0)}=2$ are assumed to be non-zero real numbers. The term "normal form" of the saddle-node bifurcation is what the equation (11) is referring to when it makes its reference to the phenomenon. This gives a substantial advantage when trying to establish the bifurcation that a system goes through, since it helps to narrow down the possible outcomes.

Consider the one-dimensional equation of the form

$$f(x,\rho)=x^3-6x^2-(\rho-10)x+\rho-5 \qquad (12)$$

Put an ρ in front of it to parameterize it. The steady-state values for (12) are found by solving for $f(x,\rho)=0$

$$\Rightarrow x^3-6x^2-(\rho-10)x+\rho-5=0$$

$$\Rightarrow x^3-6x^2-\rho x+10x+\rho-5=0$$

$$\Rightarrow x^3-x^2-5x^2+5x-\rho x+\rho+5x-5=0$$

$$\Rightarrow x^2(x-1)-5x(x-1)-\rho(x-1)+5(x-1)=0$$

$$\Rightarrow (x-1)(x^2-5x-\rho+5)=0$$

Either $(x-1)=0$ or $(x^2-5x-\rho+5)=0$

$$\Rightarrow x=1 \text{ or } x^2-5x-(\rho-5)=0$$

$$\Rightarrow x=\frac{5\pm\sqrt{25+4(\rho-5)}}{2}$$

$$\Rightarrow x=\frac{5\pm\sqrt{25+4\rho-20}}{2}$$

$$\Rightarrow x=\frac{5\pm\sqrt{5+4\rho}}{2}$$

$$\Rightarrow x=\frac{1}{2}\left(5\pm\sqrt{5+4\rho}\right) \qquad (13)$$

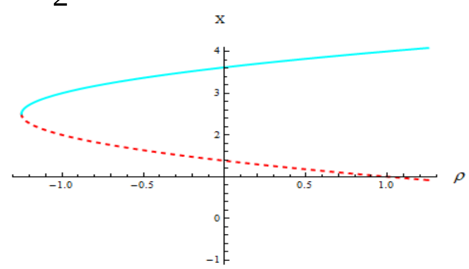


Fig. 3 Bifurcation diagram for a saddle node bifurcation of the equation (13).

Now, differentiation (12) with respect to *x*, we have

$$\left[\frac{\partial f}{\partial x}\right](x,\rho)=3x^2-12x+10-\rho$$

Again, differentiating (12) with respect to ρ, we get

$$\left[\frac{\partial f}{\partial \rho}\right](x,\rho)=-x+1$$

$$\left[\frac{\partial^2 f}{\partial x^2}\right](x,\rho)=6x-12.$$

It is seen that $f(x_0,\rho_0)=0$

$$\left[\frac{\partial f}{\partial x}\right](x_0,\rho_0)=0$$

$$\left[\frac{\partial f}{\partial \rho}\right](x_0,\rho_0) = -1 \neq 0$$

$$\left[\frac{\partial^2 f}{\partial x^2}\right](x_0,\rho_0) = 6\times 2.5 - 12 = 3 \neq 0.$$

The system (13) has a saddle-node bifurcation at (x_0, y_o). Here $x_0 = 2.5$ and $\rho_0 = -\frac{5}{4}$.

Regardless of what ρ.'s value is, it is abundantly obvious that the system has a fixed point that corresponds to x. When $\rho > 0$, the other two fixed points are represented by the equation $\Rightarrow x = \frac{1}{2}\left(5 \pm \sqrt{5+4\rho}\right)$, and when ρ is greater than zero, they are both real and distinct. When ρ is bigger than zero, they vanish, and the only time they return again is when $\rho_0 = -\frac{5}{4}$, when they correlate with the fixed point $x = 2.5$. As a consequence of this, the system experiences a saddle-node bifurcation at the point where $x = 2.5$, with $\rho_0 = -\frac{5}{4}$ acting as the point at which the system splits.

2.1(b) Regarding the equation (12), the saddle-node bifurcation can be written in its normal form.

In the event that we presume the system (1) has a point of equilibrium at $x = x_0$ for $\rho = \rho_0$ in a region where the saddle-node bifurcation requirements are satisfied from the equation (2), (3), (4), and (5), then this would suggest that the point of equilibrium is situated at a saddle-node. The result that we get when we extend the function $f(x,\rho)$ in a Taylor series in the area surrounding the point (x_0, ρ_0) is as follows:

$$\dot{x} = f(x,\rho)$$

$$= f(x_0,\rho_0) + (x - x_0)\left[\frac{\partial f}{\partial x}\right]_{(x_0,\rho_0)} + (\rho - \rho_0)\left[\frac{\partial f}{\partial \rho}\right]_{(x_0,\rho_0)} + \frac{1}{2!}(x - x_0)^2\left[\frac{\partial^2 f}{\partial x^2}\right]_{(x_0,\rho_0)}$$

$$+ (x - x_0)(\rho - \rho_0)\left[\frac{\partial^2 f}{\partial x \partial \rho}\right]_{(x_0,\rho_0)} + \frac{1}{2!}(\rho - \rho_0)^2\left[\frac{\partial^2 f}{\partial \rho^2}\right]_{(x_0,\rho_0)} + \ldots$$

Using the conditions of (2), (3), (4), and (5), the above equation contains

$$\dot{x} = f(x,\rho)$$

$$= (\rho - \rho_0)\left[\frac{\partial f}{\partial \rho}\right]_{(x_0,\rho_0)} + \frac{1}{2!}(x - x_0)^2\left[\frac{\partial^2 f}{\partial x^2}\right]_{(x_0,\rho_0)} + \ldots \tag{14}$$

From the system (12)

$$f(x,\rho) = x^3 - 6x^2 - (\rho - 10)x + \rho - 5;\; x \in \Re \tag{15}$$

Differentiating (15) with respect to ρ

$\left[\frac{\partial f}{\partial \rho}\right]_{(x_0,\rho_0)} = -x + 1$ [Taking $x_0 = 2.5$ and $\rho_0 = -\frac{5}{4}$]

$$\left[\frac{\partial f}{\partial \rho}\right]_{\left(2.5,-\frac{5}{4}\right)} = -2.5 + 1 = -1.5$$

Differentiating (15) with respect to x

$$\left[\frac{\partial f}{\partial x}\right]_{(x_0,\rho_0)} = 3x^2 - 12x - \rho + 10 \tag{16}$$

[Taking $x_0 = 2.5$ and $\rho_0 = -\frac{5}{4}$]

$$\left[\frac{\partial f}{\partial x}\right]_{\left(2.5,-\frac{5}{4}\right)} = 3\times(2.5)^2 - 12\times 2.5 - \left(-\frac{5}{4}\right) + 10$$

$$\left[\frac{\partial f}{\partial x}\right]_{\left(2.5,-\frac{5}{4}\right)} = 18.75 - 30 - \left(-1.25\right) + 10$$

$$\left[\frac{\partial f}{\partial x}\right]_{\left(2.5,-\frac{5}{4}\right)} = 18.75 - 30 + 1.25 + 10$$

$$\left[\frac{\partial f}{\partial x}\right]_{\left(2.5,-\frac{5}{4}\right)} = 30 - 30 = 0$$

Differentiating (16) with respect to x

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{(x_0,\rho_0)} = 6x - 12$$

[Taking $x_0 = 2.5$ and $\rho_0 = -\frac{5}{4}$]

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(2.5,-\frac{5}{4}\right)} = 6\times 2.5 - 12 = 15 - 12 = 3$$

Putting the above values from (14) we get,

$$\dot{x} = f\left(x, \rho\right)$$

$$= \left(\rho + \frac{5}{4}\right)\left[\frac{\partial f}{\partial \rho}\right]_{\left(2.5,-\frac{5}{4}\right)} + \frac{1}{2!}\left(x - 2.5\right)^2 \left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(2.5,-\frac{5}{4}\right)} + \ldots$$

$$= \left(\rho + \frac{5}{4}\right).\left(-1.5\right) + \frac{1}{2!}\left(x - 2.5\right)^2 \times 3 + \ldots$$

$$= \left(\rho + \frac{5}{4}\right).\phi + \frac{1}{2!}\left(x - 2.5\right)^2 \times \varphi + \ldots \quad (17)$$

In this equation, both $\phi = \left[\frac{\partial f}{\partial \rho}\right]_{\left(2.5,-\frac{5}{4}\right)} = -1.5$ and $\varphi = \left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(2.5,-\frac{5}{4}\right)} = 3$ are assumed to be non-zero real numbers. The term "normal form" of the saddle-node bifurcation is what the equation (17) is referring to when it makes its reference to the phenomenon. This gives a substantial advantage when trying to establish the bifurcation that a system goes through, since it helps to narrow down the possible outcomes.

3. Transcritical bifurcation

There is a subcategory of bifurcation theory called transcritical bifurcation, which describes the situation in which there is equilibrium with an eigenvalue in which the real component crosses zero. At a transcritical bifurcation, there is always a fixed point present, regardless of the values of the parameters. However, as the parameter is altered, so does the fixed point's stability. Both the unstable and stable fixed points are present before and after the bifurcation. The stability of one decreases and that of the other increases as they collide. This causes the previously unstable fixed point to become stable. After that, it can take either of two secure (stable)or risky(unstable) paths. This transition is called a transcritical bifurcation.

There are a variety of sub-categories available of physical systems that are dependent on parameters, and one of the kinds of systems that fall into this category is one in which an equilibrium point is required to exist for all possible values of a parameter of the system and can never cease to exist. Other types of physical systems that are reliant on parameters include those that are dependent on variables. Another form of system that is dependent on parameters is one that consists of numerous physical systems, each of which is dependent on the system's parameters. However, depending on the value of the parameter, the stability's behavior may shift. This is because the parameter's value changes over time. This is because the parameter's value is highly unpredictable. Because the stable features of the fixed points vary with the values of the parameters, transcritical bifurcation may be distinguished from other types of bifurcation. Subcritical bifurcations are similar to transcritical bifurcations, which are still another type of bifurcation. The term "transcritical bifurcation" was originally developed to characterize this observable phenomenon.

Take the parameterized C^2 map family $f : R\times R \to R$ as an example.

$$f\left(x_0, \rho_0\right) = 0 \quad (18)$$

$$\left[\frac{\partial f}{\partial x}\right]_{(x_0,\rho_0)} = 0 \quad (19)$$

$$\left[\frac{\partial f}{\partial \rho}\right]_{(x_0,\rho_0)}=0 \tag{20}$$

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{(x_0,\rho_0)}\neq 0 \tag{21}$$

$$\left[\frac{\partial^2 f}{\partial x \partial \rho}\right]_{(x_0,\rho_0)}\neq 0 \tag{22}$$

The following equation illustrates a transcritical bifurcation (Guckenheimer and Holmes 1997, p. 145):

$$f\left(x,\rho\right)=\rho x-x^2+x \tag{23}$$

$\rho\in\Re$ are serving as the input parameter. Consideration is given to the system's equilibrium points as

$$f\left(x,\rho\right)=0$$

$$\Rightarrow \rho x-x^2+x=0$$

$$\Rightarrow x\left(\rho-x+1\right)=0$$

Either $x=0$ or $\rho-x+1=0$

$$\Rightarrow -x=-1-\rho$$

$$\Rightarrow x=1+\rho$$

This means there are two stable configurations, $x^{*}=0,\rho+1.$ Our Estimates

$$\left[\frac{\partial f}{\partial x}\right]\left(x,\rho\right)=\rho-2x+1$$

$$\Rightarrow \left[\frac{\partial f}{\partial x}\right]\left(0,\rho\right)=\rho+1$$

$$\therefore \left[\frac{\partial f}{\partial x}\right]\left(\rho,\rho\right)=\rho-2\rho+1$$

$$\Rightarrow \left[\frac{\partial f}{\partial x}\right]\left(\rho,\rho\right)=-\rho+1$$

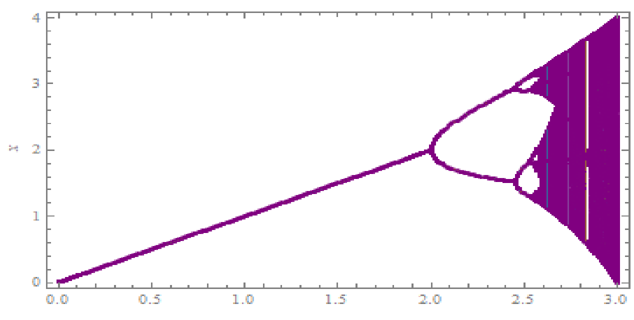
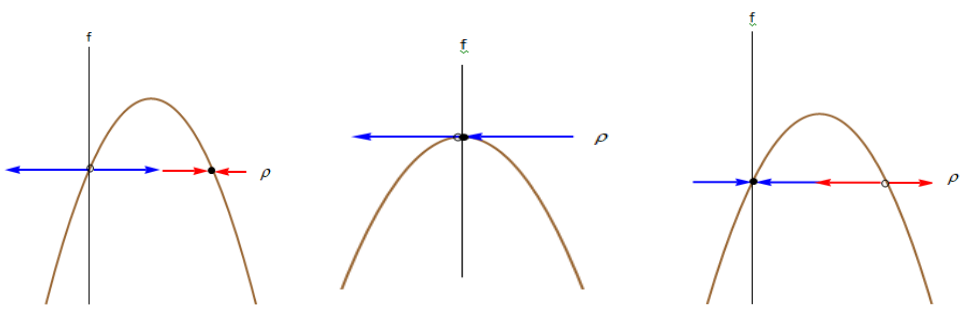


Fig. 4 A transcritical bifurcation of the equation (23) by orbit diagram for arbitrary value of x and ρ.



$(a)\,\rho>0$ $(b)\,\rho=0$ $(c)\,\rho<0$

Fig. 5 Phase diagram for (a) $\rho>0,$ (b) $\rho=0,$ and (c) $\rho<0$ of the equation (23).

The origin, which is a source (unstable) for $\rho > 0$ and a sink (stable) for $\rho < 0$, forms two separate equilibrium points $x^* = 0, \rho + 1$ for $\rho \neq 0$. If ρ is greater than zero, the opposite equilibrium point, $x^* = \rho + 1$ is unstable and stable for $\rho > 0$. Figure 5 depicts the phase diagrams for the aforementioned three scenarios.

3.1 (a) Regarding the equation (23), the transcritical bifurcation can be written in its normal form.

In the event that we presume the system (1) has a point of equilibrium at $x = x_0$ for $\rho = \rho_0$ in a region where the transcritical bifurcation requirements are satisfied from the equation (18), (19), (20), (21), and (22) then this would suggest that the point of equilibrium is situated at a trancritical. The result that we get when we extend the function $f(x, \rho)$ in a Taylor series in the area surrounding the point (x_0, ρ_0) is as follows:

$$\dot{x} = f(x, \rho)$$

$$= f(x_0, \rho_0) + (x - x_0)\left[\frac{\partial f}{\partial x}\right]_{(x_0, \rho_0)} + (\rho - \rho_0)\left[\frac{\partial f}{\partial \rho}\right]_{(x_0, \rho_0)} + \frac{1}{2!}(x - x_0)^2\left[\frac{\partial^2 f}{\partial x^2}\right]_{(x_0, \rho_0)}$$

$$+ (x - x_0)(\rho - \rho_0)\left[\frac{\partial^2 f}{\partial x \partial \rho}\right]_{(x_0, \rho_0)} + \frac{1}{2!}(\rho - \rho_0)^2\left[\frac{\partial^2 f}{\partial \rho^2}\right]_{(x_0, \rho_0)} + \ldots$$

Using the conditions of (18), (19), (20), (21), and (22) the above equation contains

$$\dot{x} = f(x, \rho)$$

$$= (x - x_0)(\rho - \rho_0)\left[\frac{\partial^2 f}{\partial x \partial \rho}\right](x_0, \rho_0) + \frac{1}{2!}(x - x_0)^2\left[\frac{\partial^2 f}{\partial x^2}\right](x_0, \rho_0) + \ldots \quad (24)$$

From the system (23)

$$f(x, \rho) = \rho x - x^2 + x;\ x \in \Re \quad (25)$$

Differentiating (25) with respect to ρ

$$\left[\frac{\partial f}{\partial \rho}\right]_{(x_0, \rho_0)} = x$$

$$\left[\frac{\partial f}{\partial \rho}\right]_{(x_0, \rho_0)} = 0 \text{ [Taking } x_0 = 0 \text{ and } \rho = -1]$$

Differentiating (25) with respect to x

$$\left[\frac{\partial f}{\partial x}\right](x_0, \rho_0) = \rho - 2x + 1 \quad (26)$$

[Taking $x_0 = 0$ and $\rho = -1$]

$$\left[\frac{\partial f}{\partial x}\right](0, -1) = \rho - 2x + 1 = -1 - 2.0 + 1 = 0$$

Differentiating (26) with respect to x

$$\left[\frac{\partial^2 f}{\partial x^2}\right](x_0, \rho_0) = -2$$

[Taking $x_0 = 0$ and $\rho = -1$]

$$\left[\frac{\partial^2 f}{\partial x^2}\right](0, -1) = -2$$

$$\left[\frac{\partial^2 f}{\partial x \partial \rho}\right](x_0, \rho_0) = 1$$

$$\left[\frac{\partial^2 f}{\partial x \partial \rho}\right](0, -1) = 1$$

Putting the above values from (24) we get,

$$\dot{x} = f(x, \rho)$$

$$= (x - 0)(\rho + 1)\left[\frac{\partial^2 f}{\partial x \partial \rho}\right](0, -1) + \frac{1}{2!}(x - 0)^2\left[\frac{\partial^2 f}{\partial x^2}\right](0, -1) + \ldots$$

$$= x(\rho + 1) \times 1 + \frac{1}{2!}x^2(-2) + \ldots$$

$$= x(\rho + 1) \times \phi + \frac{1}{2!}x^2 \varphi + \ldots \quad (27)$$

In this equation, both $\phi = \left[\frac{\partial^2 f}{\partial x \partial \rho}\right](0,-1) = 1$ and $\varphi = \left[\frac{\partial^2 f}{\partial x^2}\right](0,-1) = -2$ are assumed to be non-zero real numbers. The term "normal form" of the transcritical bifurcation is what the equation (27) is referring to when it makes its reference to the phenomenon. This gives a substantial advantage when trying to establish the bifurcation that a system goes through, since it helps to narrow down the possible outcomes.

4. Pitchfork bifurcation

The transition from a single fixed point to three fixed points is a case of a local bifurcation referred as a pitchfork bifurcation. Pitchfork bifurcations are available in supercritical and subcritical varieties, just as Hopf bifurcations do. In persistent dynamical systems, such as flows, modeled by ordinary differential equations, symmetry destruction may appear in the form of pitchfork bifurcations.

Take the parameterized C^3 map family $f : R \times R \to R$ as an example.

$$f(-x,\rho) = -f(x,\rho) \tag{28}$$

$$f(x_0,\rho_0) = 0 \tag{29}$$

$$\left[\frac{\partial f}{\partial x}\right]_{(x_0,\rho_0)} = 0 \tag{30}$$

$$\left[\frac{\partial f}{\partial \rho}\right]_{(x_0,\rho_0)} = 0 \tag{31}$$

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{(x_0,\rho_0)} = 0 \tag{32}$$

$$\left[\frac{\partial^2 f}{\partial x \partial \rho}\right]_{(x_0,\rho_0)} \neq 0 \tag{33}$$

$$\left[\frac{\partial^3 f}{\partial x^3}\right]_{(x_0,\rho_0)} \neq 0 \tag{34}$$

However, (35) can have some leeway. (Rasband 1990, p. 31). Different kinds of intervals include those with two and three points. (Two of which are stable and one of which is unstable). This type of split is known as a "pitchfork bifurcation."The equation exhibits a pitchfork bifurcation. Pitchfork bifurcation is a type of bifurcation that occurs in a system with only one dimension when the system has left-right symmetry. In this kind of system, fixed points frequently come into existence or vanish in pairs. Take, for example, a system that only consists of one dimension

$$f(x,\rho) = \rho x - x^3 + x \tag{35}$$

Replacing x by $-x$ in (35), it contains that

$$f(-x,\rho) = -\rho x + x^3 - x$$
$$= -\left(\rho x - x^3 + x\right)$$
$$= -f(x,\rho)$$

For this reason, applying transformation $f(-x,\rho) = -f(x,\rho)$ to the system has no effect on its overall geometry. The equilibrium points of the system can be found by using the following formula:

$$f(x,\rho) = 0$$
$$\Rightarrow \rho x - x^3 + x = 0$$
$$\Rightarrow x\left(\rho - x^2 + 1\right) = 0$$

Either $x = 0$ or $\rho - x^2 + 1 = 0$

$$\Rightarrow -x^2 = -1 - \rho$$
$$\Rightarrow x^2 = 1 + \rho$$
$$\Rightarrow x = \pm\sqrt{1+\rho}$$

Now $f(x,\rho) = \rho x - x^3 + x$

$$\left[\frac{\partial f}{\partial x}\right]_{x,\rho} = \rho - 3x^2 + 1$$

$$\left[\frac{\partial f}{\partial x}\right]_{0,\rho} = \rho + 1$$

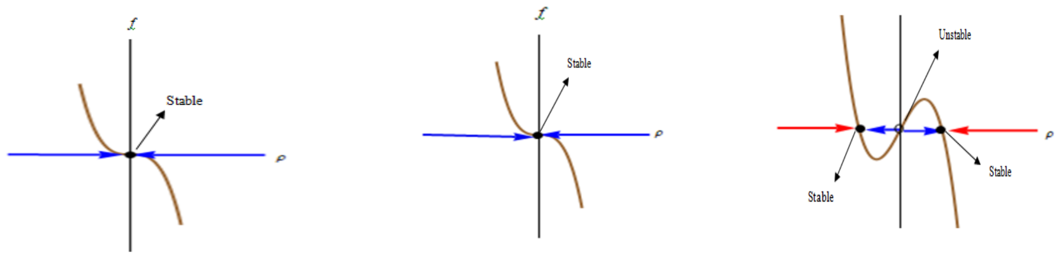
$$\left[\frac{\partial f}{\partial x}\right]_{(\pm\sqrt{\rho+1},\rho)} = \rho - 3\left(\pm\sqrt{\rho+1}\right)^2 + 1$$

$$\Rightarrow \left[\frac{\partial f}{\partial x}\right]_{(\pm\sqrt{\rho+1},\rho)} = \rho - 3(\rho+1) + 1$$

$$\Rightarrow \left[\frac{\partial f}{\partial x}\right]_{(\pm\sqrt{\rho+1},\rho)} = \rho - 3\rho - 3 + 1$$

$$\Rightarrow \left[\frac{\partial f}{\partial x}\right]_{(\pm\sqrt{\rho+1},\rho)} = -2\rho - 2$$

$$\Rightarrow \left[\frac{\partial f}{\partial x}\right]_{(\pm\sqrt{\rho+1},\rho)} = -2(\rho+1)$$



$(a)\,\rho<0$ $(b)\,\rho=0$ $(c)\,\rho>0$

Fig. 6 Phase diagram for (a) $\rho<0$, (b) $\rho=0$, and (c) $\rho>0$ of the equation (35).

When $\rho=0$, x^* only equals zero at one point in the system, which also happens to be where $\left[\frac{\partial f}{\partial x}\right]_{(x=0,\rho=0)}=0$ in nature. If $\rho>0$, then there will be three equilibrium points at $x^*=0,\pm\sqrt{\rho+1}$. The origin of the equilibrium points ($x^*=0$) is an unstable source, while the other two equilibrium points are stable sinks. If $\rho<0$, there is only one stable equilibrium point for the system, and it is at the origin. Figure 6depicts the phase diagram in the $\rho-f$ plane.

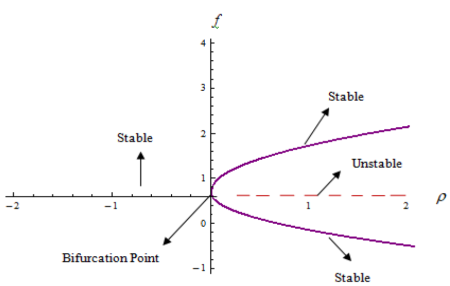


Fig. 7 Pitchfork supercritical bifurcation for the one-dimensional system (35).

Supercritical case

The fundamental outline of the supercritical pitchfork can be described in the following manner, which is provided by the symbol for the third derivative in the equation containing the number (36).

$$\left[\frac{\partial^3 f}{\partial x^3}\right]_{\rho=0,x=0} < 0 \qquad (36)$$

When ρ goes from being negative to being zero, the diagram shows that the equilibrium point origin remains stable, albeit in a manner that is significantly less robust due to the fact that it is not hyperbolic. This is something that we can see for ourselves by looking at the diagram. When ρ is greater than zero, the origin, which was formerly a point of stable equilibrium, changes into a point of unstable equilibrium, and two new points of stable equilibrium appear on either side of the origin, at $x=-\sqrt{\rho+1}$ and $x=\sqrt{\rho+1}$, respectively.

When ρ less than zero, the origin remains a point of stable equilibrium. To take a look at the bifurcation diagram of the system that is depicted in Figure 7. The pitchfork-shape bifurcation diagram elucidates the meaning of the word "pitchfork" in a way that is both concise and comprehensive. On the other hand, one could consider it to be a bifurcation of the system similar to a pitchfork. It is the existence of this vector field that causes this bifurcation to occur, and it is commonly referred to as a supercritical pitchfork bifurcation. This type of bifurcation takes place when an equilibrium that is stable breaks up into two separate equilibriums that are also stable.

Subcritical case

The fundamental outline of the subcritical pitchfork can be described in the following manner, which is provided by the symbol for the third derivative in the equation containing the number (37).

$$\left[\frac{\partial^3 f}{\partial x^3}\right]_{\rho=0,x=0} > 0 \tag{37}$$

Take, for example, a system that only consists of one dimension

$$f(x,\rho) = \rho x + x^3 + x \tag{38}$$

Replacing *x* by –*x* in (38), it contains that

$$f(-x,\rho) = -\rho x - x^3 - x$$

$$= -(\rho x + x^3 + x)$$

$$= -f(x,\rho)$$

For this reason, applying transformation $f(-x,\rho) = -f(x,\rho)$ to the system has no effect on its overall geometry. The equilibrium points of the system can be found by using the following formula:

$$f(x,\rho) = 0$$

$$\Rightarrow \rho x + x^3 + x = 0$$

$$\Rightarrow x(\rho + x^2 + 1) = 0$$

Either $x = 0$ or $\rho + x^2 + 1 = 0$

$$\Rightarrow x^2 = -1 - \rho$$

$$\Rightarrow x^2 = -1 - \rho$$

$$\Rightarrow x = \pm\sqrt{-1-\rho}$$

Now $f(x,\rho) = \rho x + x^3 + x$

$$\left[\frac{\partial f}{\partial x}\right]_{x,\rho} = \rho + 3x^2 + 1$$

$$\left[\frac{\partial f}{\partial x}\right]_{0,\rho} = \rho + 1$$

$$\left[\frac{\partial f}{\partial x}\right]_{(\pm\sqrt{-\rho-1},\rho)} = \rho - 3\left(\pm\sqrt{-\rho-1}\right)^2 + 1$$

$$\Rightarrow \left[\frac{\partial f}{\partial x}\right]_{(\pm\sqrt{-\rho-1},\rho)} = \rho - 3(-\rho-1) + 1$$

$$\Rightarrow \left[\frac{\partial f}{\partial x}\right]_{(\pm\sqrt{-\rho-1},\rho)} = \rho + 3\rho + 3 + 1$$

$$\Rightarrow \left[\frac{\partial f}{\partial x}\right]_{(\pm\sqrt{-\rho-1},\rho)} = 4\rho + 4$$

$$\Rightarrow \left[\frac{\partial f}{\partial x}\right]_{(\pm\sqrt{-\rho-1},\rho)} = 4(\rho + 1)$$

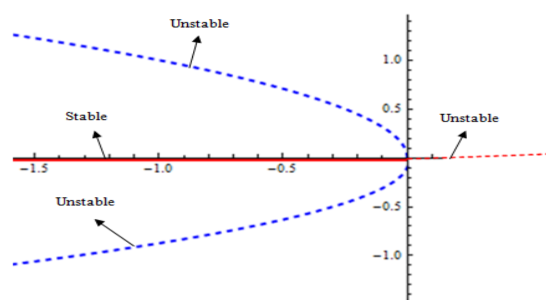


Fig. 8 Pitchfork subcritical bifurcation for the one-dimensional system (35).

This system has three equilibrium points, which are denoted by the symbols $x^* = 0, \pm\sqrt{-\rho-1}$ for $\rho < 0$. The equilibrium point $x^* = 0$ is stable, whereas the other two equilibrium points are unstable. When p is greater than zero, there is only one possible point of equilibrium $x^* = 0$, and it is an unstable one.

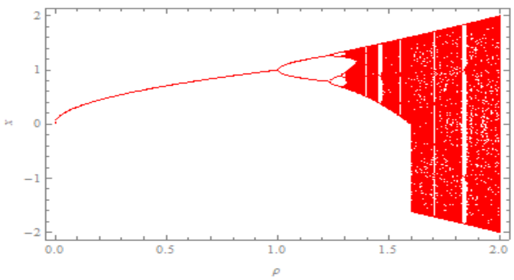


Fig. 9 A pitchfork bifurcation of the equation (35) by orbit diagram for arbitrary value of x and ρ.

4.1(a) Regarding the equation (35), the pitchfork bifurcation can be written in its normal form.

In the event that we presume the system (1) has a point of equilibrium at $x = x_0$ for $\rho = \rho_0$ in a region where the pitchfork bifurcation requirements are satisfied from the equation (28), (29), (30), (31), (32) (33), and (34) then this would suggest that the point of equilibrium is situated at a pitchfork. The result that we get when we extend the function $f(x,\rho)$ in a Taylor series in the area surrounding the point (x_0, ρ_0) is as follows:

$$\dot{x} = f(x,\rho)$$

$$= f(x_0,\rho_0) + (x - x_0)\left[\frac{\partial f}{\partial x}\right]_{(x_0,\rho_0)} + (\rho-\rho_0)\left[\frac{\partial f}{\partial \rho}\right]_{(x_0,\rho_0)} + \frac{1}{2!}(x-x_0)^2\left[\frac{\partial^2 f}{\partial x^2}\right](x_0,\rho_0)$$

$$(x-x_0)(\rho-\rho_0)\left[\frac{\partial^2 f}{\partial x \partial \mu}\right]_{(x_0,\rho_0)} + \frac{1}{2!}(\rho-\rho_0)^2\left[\frac{\partial^2 f}{\partial \rho^2}\right](x_0,\rho_0) +$$

$$\frac{1}{6}(x-x_0)^3\left[\frac{\partial^3 f}{\partial x^3}\right]_{(x_0,\rho_0)} + \frac{1}{2}(x-x_0)^2(\rho-\rho_0)\left[\frac{\partial^3 f}{\partial x \partial \mu}\right]_{(x_0,\rho_0)} +$$

$$\frac{1}{2}(x-x_0)\ (\rho-\rho_0)^2\left[\frac{\partial^3 f}{\partial x \partial \mu^2}\right]_{(x_0,\rho_0)} + \frac{1}{6}(\rho-\rho_0)^3\left[\frac{\partial^2 f}{\partial \rho^2}\right]_{(x_0,\rho_0)} + \ldots$$

Using the conditions of (28), (29), (30), (31), (32) (33), and (34) the above equation contains

$$\dot{x} = f(x,\rho)$$

$$= (x-x_0)(\rho-\rho_0)\left[\frac{\partial^2 f}{\partial x \partial \rho}\right](x_0,\rho_0) + \frac{1}{6}(x-x_0)^3\left[\frac{\partial^3 f}{\partial x^3}\right](x_0,\rho_0) + \ldots \qquad (39)$$

From the system (35)

$$f(x,\rho) = \rho x - x^3 + x;\ x \in \Re \qquad (40)$$

Differentiating (40) with respect to x

$$\left[\frac{\partial f}{\partial x}\right](x_0,\rho_0) = \rho - 3x^2 + 1 \qquad (41)$$

Differentiating (41) with respect to x

$$\left[\frac{\partial^2 f}{\partial x^2}\right](x_0,\rho_0) = -6x$$

$$\left[\frac{\partial^3 f}{\partial x^3}\right](x_0,\rho_0) = -6$$

[Taking $x_0 = 0$ and $\rho = -1$]

$$\left[\frac{\partial^3 f}{\partial x^3}\right](0,-1) = -6$$

Differentiating (41) with respect to ρ

$$\left[\frac{\partial^2 f}{\partial x \partial \rho}\right](x_0,\rho_0) = 1$$

[Taking $x_0 = 0$ and $\rho_0 = -1$]

$$\left[\frac{\partial^2 f}{\partial x \partial \rho}\right](0,-1) = 1$$

Putting the above values from (39) we get,

$$\dot{x} = f(x,\rho)$$

$$= (x-0)(\rho+1)\left[\frac{\partial^2 f}{\partial x \partial \rho}\right](0,-1) + \frac{1}{6}(x-0)^3\left[\frac{\partial^3 f}{\partial x^3}\right](0,-1) + ...$$

$$= x(\rho+1)\times 1 + \frac{1}{6}x^3(-6) + ...$$

$$= x(\rho+1)\times\phi + \frac{1}{6}x^3\varphi + ... \qquad (42)$$

In this equation, both $\phi = \left[\frac{\partial^2 f}{\partial x \partial \rho}\right](0,-1) = 1$ and $\varphi = \left[\frac{\partial^3 f}{\partial x^3}\right](0,-1) = -6$ are assumed to be non-zero real numbers. The term "normal form" of the pitchfork bifurcation is what the equation (42) is referring to when it makes its reference to the phenomenon. This gives a substantial advantage when trying to establish the bifurcation that a system goes through, since it helps to narrow down the possible outcomes.

5. Conclusion

For saddle node bifurcation, depending on the value that assigns to the parameter ρ, there are three possible outcomes that are capable of taking place. When ρ is non-zero, the system has two fixed positions that it can always return to. If ρ is less than zero, then they are combined together; however, if ρ is more than zero, then they are eliminated. The dynamics of such a fundamental system are fascinating despite their seeming simplicity. The dynamics underwent a bifurcation at $\rho = 0$ due to the inherent incompatibility of the $\rho < 0$ and $\rho > 0$ vector fields. The bifurcation point, or turning point of the trajectory, is the location along the trajectory at which the parameter ρ equals 0. In the bifurcation diagram, we can see how ρ is related to the fixed point of the system.During transcritical bifurcation, the origin, which is a sink (stable) for $\rho = 0$ and a source (unstable) for $\rho > 0$, generates two separate equilibrium positions $x = 0, \rho+1$ for $\rho \neq 0$. In contrast, the equilibrium at $x = \rho+1$ is unstable when $\rho > 0$ but stable for $\rho = 0$. If $\rho > 0$, then pitchfork bifurcation occurs, where there are three equilibrium points $x^* = 0, \pm\sqrt{\rho+1}$. In contrast to the two sinks at the other equilibrium points, the origin ($x^* = 0$) is a source of potential instability. The only location where equilibrium may be achieved when ρ is less than zero is the origin.As shown by the pitchfork bifurcation diagram, the term "pitchfork" can be defined both succinctly and exhaustively. It's possible, though, that the system is bifurcating in a pitchfork shape. Pitchfork bifurcations at the subercritical point are triggered by this vector field. For $\rho < 0$ the equilibrium point $x^* = 0$ is stable, whereas the other two equilibrium points are unstable.

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