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Performance Analysis of Trinomial Tree Model for European and American Option Pricing

Research Article

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ABSTRACT

In financial market, option pricing plays a vital role in the stock market and having significant outlook in the economic development of a nation. In business and stock market, different financial models are developed for computing options for stocks or bonds. In this work, we focused on basic concepts of trinomial tree model, derivation and algorithm and application of it. Here, we apply the trinomial model for American and European option pricing. Basically, we applied the techniques to obtain American and European option prices to incorporate an investor's view of the future behavior of the stock market related to European and American options. Finally, we showed some basic difference between American and European option applying trinomial model.

Keywords: *Binominal model, Trinomial model, European options, American options*

Abbreviation:

BM: Binomial Model; BSM: Black Scholes Model; TM: Trinomial Model

1. Introduction

Option pricing (call/put) is one of the most important aspects of trading the derivatives. With the founding of the Chicago Board option exchange in April 1997, options have emerged as the most dynamic component of the stock market (Black et.al. 1973). In the case of a call option, it grants the holder the right, but not the obligation, to purchase the underlying asset at a preset price (exercise or strike price) on the maturity date (expiry date). Similarly, for put options, it grants the holder the right, but not the obligation, to sell the underlying asset at a preset price (exercise or strike price) on the maturity date (Merton 1973; Capinski et al. 2011).

Black et al. introduced the first model of fully

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balanced option pricing (Black et. al. 1973) named as Black-Schole model. Also, Robert Merton (Merton 1973) offers several interpretations of the Black-Scholes Model (BSM). Since its introduction to the market in 1973, BSM has been extensively utilized by traders to determine the option's price Cox et. al. (Cox et. al. 1976) demonstrated that underlying assets' prices fluctuate rather than move constantly. Cox et. al. (Cox et al. 1979) introduced the Binomial Tree Model (BM) for option calculating in 1979, which was another option model available in the market. BM is a straightforward and simple concept. Feng et al. (Feng et.al. 2012) showed that the BM converges to the well-known BSM as the number of binomial periods increases. Phelim Boyle invented the Monte Carlo simulation technique in 1977. Then for more price fluctuating rates the Trinomial tree model (TM) was developed by Phelim Boyle in 1986 (Boyle 1986, 1988).The Trinomial tree model provides a flexible framework for valuing options and other derivatives, particularly when the underlying asset's price movements are not easily modeled by a continuous process or when discrete factors significantly impact price dynamics. It is a lattice based computational model used in financial mathematics to price options. It is an extension of the binomial options pricing model, and is conceptually similar (Ingersoll et al. 1976; Jalaludin et al. 2024; Ratibenyakool et al. 2022).

In this work, we focused on basic concepts of trinomial tree model, derivation, algorithm and application of it. Basically, we discussed the technique to obtain American and European option prices which shows the better performance comparing to BM. Also, we have shown the difference between BM and TM with probability in some examples. In this work, we have shown that trinomial model is a good choice in the case of price fluctuating situation. Applying in some real cases (Yunzhang et. al. 2023; Lok et. al. 2022; Bonner et al. 2024), we observed that the Trinomial tree model provides a flexible framework for valuing options and other derivatives, individually when some underlying asset’s price movements are not following the continuous process.

The rest of the paper is organized as follows: In Section2, we have discussed the fundamental concepts for trinomial tree. Section 3 focuses on derivation and implementation of trinomial model. Also, some real renowned option pricing problems, like American and European put options are solved using trinomial model which can be found in Section 4. Finally, we have drawn the conclusion and necessary references at the end of the project paper.

2.1 Trinomial Tree Model

The trinomial model is a mathematical model generally used in finance for option pricing of an risky asset and other derivative securities in the world (Yunzhang et al. 2023). It is an extension of the binomial model, which itself is based on the principles of no-arbitrage and risk-neutral valuation (Cox et al.1979 ; Fama 1965). A risky asset is any asset meant a degree of risk. Risk asset generally refers to assets that have a significant degree of price volatility, such as equities, commodities, high-yield bonds, real estate and currencies. An example of a risk-free asset is a bank deposit or Treasury bond issued by the government, a financial institution or a company current and future bond value is known to investor. An asset where you already know for sure how much you'll get back is risk free asset (Capinski et al. 2011; Jalaludin et al. 2024; Ratibenyakool et al. 2022). In the trinomial model, the underlying asset's price is move in three possible directions at each time step: up, down, or remain unchanged. This contrasts with the binomial model, which only allows for two possible price movements (rise or down) at each time step (Coval et al. 2001).

Here, we firstly introduce a simple structure of trinomial tree model stock price movement; we discuss the generalization of trinomial model with probability process later. Introducing initial stock price s_0 the factor u used for upward price movement and d for downward price movement,then the simple model arrived for three step price movement as

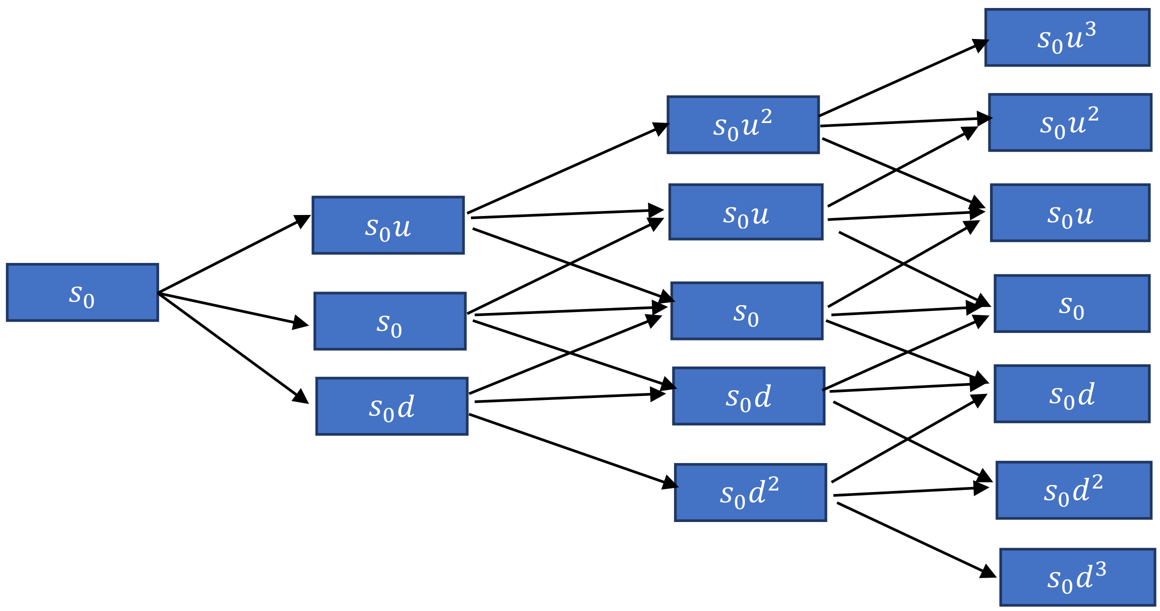

Figure 1: Trinomial Stock price

2.2Algorithm of Trinomial Model

1: Declare and initialize $S(0)$

2: Calculate the jump sizes u,d and m

3: Calculate the transition probabilitiesp_u , $p_d$and p_m

4: Build the share price tree

5: Calculate the payoff of the option at maturity, i.e. node N

6:$\boldsymbol{for}\ (\boldsymbol{int\ i} = \boldsymbol{N};\ \boldsymbol{i} > 0;\ --i)\ do$

7: Calculate option price at node$i$based on

8: $\boldsymbol{C_{n,j} = e^{-r\Delta t}\ [p_u C_{n+1,j+1} + p_m C_{n+1,j} + p_d\ C_{n+1,j-1}]}$

9: $\boldsymbol{end\ for}$

10: Output option price

Problem : Assume a stock whose current price is \$140. The stock price has three types movement upward, downward and remain flat over a certain time (Capinski et. al. 2003).

Also assume the parameters:

upward movement factor (u) = 1.2000

downward movement factor (d) = 0.8333

1st step: 140*1.20 = 168, 140*1 = 140, 140*0.83 = 116.2

In a similar manner the tree arrived as

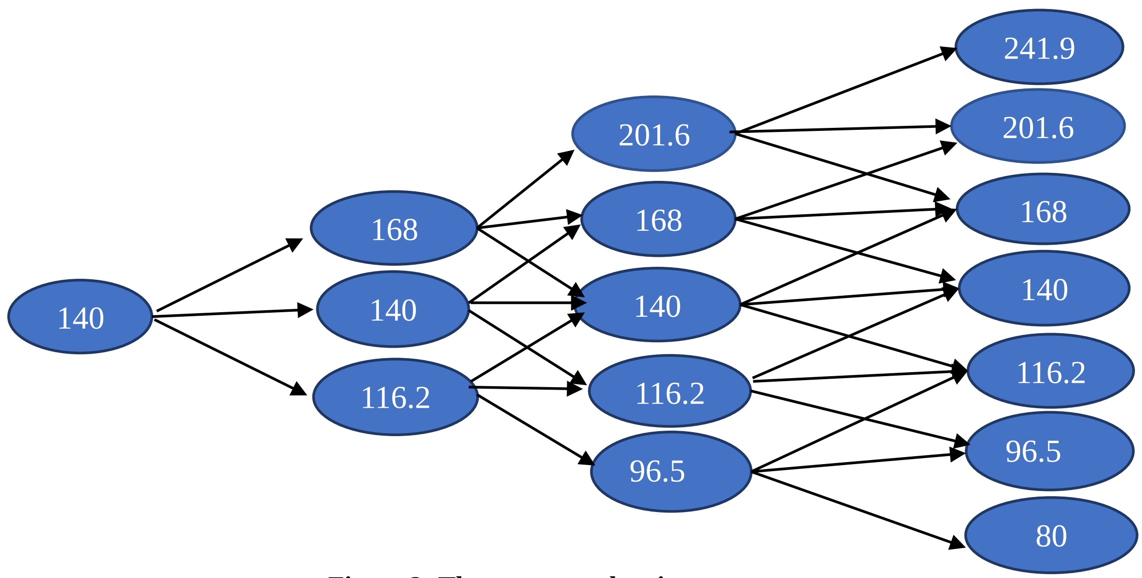

Figure 2: Three step stock price movements

3.Trinomial Tree Model with Probability Distribution

Now we will derive the trinomial tree model with its probability. The trinomial distribution is a probability distribution that generalizes the binomial distribution to three scenarios instead of two. It deals with the probabilities of the outcomes of a random experiment that can result in one of three possible outcomes, defined as success, loss and neutral with respective probabilities p_u, p_d and $p_m$where $p_u+p_d+p_m=1$.

A natural generalization (Capinski et al. and Zastawniak, et al. 2003) from binomial tree model extends to take amiddle value at any given step the range of possible values of the one-step returns K(n) are of the form

$$K(n) = \begin{cases} u \text{ with probability } p_u, \\ n\ with\ probability\ p_m, \\ d\ with\ probability\ p_d, \end{cases}$$

where d<n<u and $0<p_u, p_m, p_u+p_m<1$

This means that u and d represent upward and downward price movements, as before, whereas n stands for the intermediate price movement. Since S(1) =S(0)(1 + K(1)) implies that S(1) takes three different values,

$$S(1) = \begin{cases} S(0)(1+u) \text{ with probability } p_u, \\ S(0)(1+\text{n}) \text{ with probability } p_m, \\ S(0)(1+d)\text{with probability } p_d, \end{cases}$$

We draw a three steps trinomial model with probability p_u,$p_m$and p_d.

At time step one S(1) takes three types of movement.

Upward, neutral and downward then the price movement scenario in this structure:

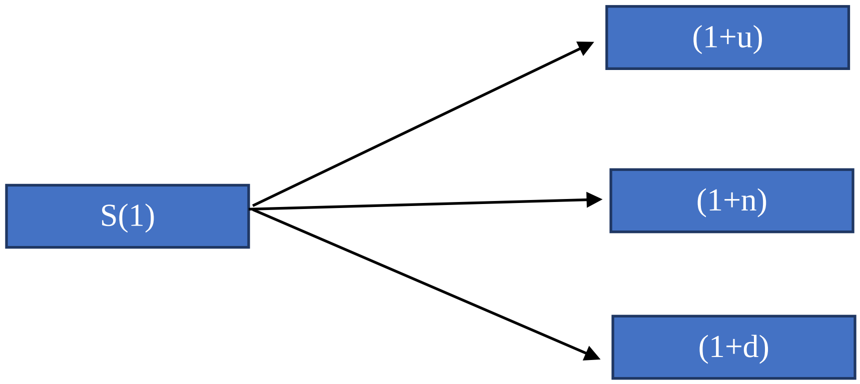

Figure 3: first step of price movement

For generalization we need to continue this process minimum three times. In the first step we get three price movement scenario, second step we get nine price movement but seven values are different one. For ease of understanding in this script we show the third step and find out the different value takes S(3).The price S(3) takes 27 values corresponding to 27 price movement scenarios. Among these 27 values there are only 10 values (But in particularly 7 values) are different one. These are:

$$S(3) = \begin{cases} S(0)(1+u)^3 & \text{with probability } {p_u}^3, \\ S(0)(1+u)^2(1+n) & \text{with probability } 3{p_u}^2 p_m, \\ S(0)(1+u)^2(1+d) & \text{with probability } 3{p_u}^2 p_d, \\ S(0)(1+u)(1+n)^2 & \text{with probability } 3p_u {p_m}^2, \\ S(0)(1+u)(1+d)^2 & \text{with probability } 3p_m {p_d}^2, \\ S(0)(1+u)(1+n)(1+d) & \text{with probability } 6p_u p_m p_d, \\ S(0)(1+n)^2(1+d) & \text{with probability } 3{p_m}^2 p_d, \\ S(0)(1+n)(1+d)^2 & \text{with probability } 3p_m {p_d}^2, \\ S(0)(1+d)^3 & \text{with probability } {p_d}^3, \\ S(0)(1+n)^3 & \text{with probability } {p_m}^3, \end{cases} \quad \ldots\ldots\ldots\ldots (1)$$

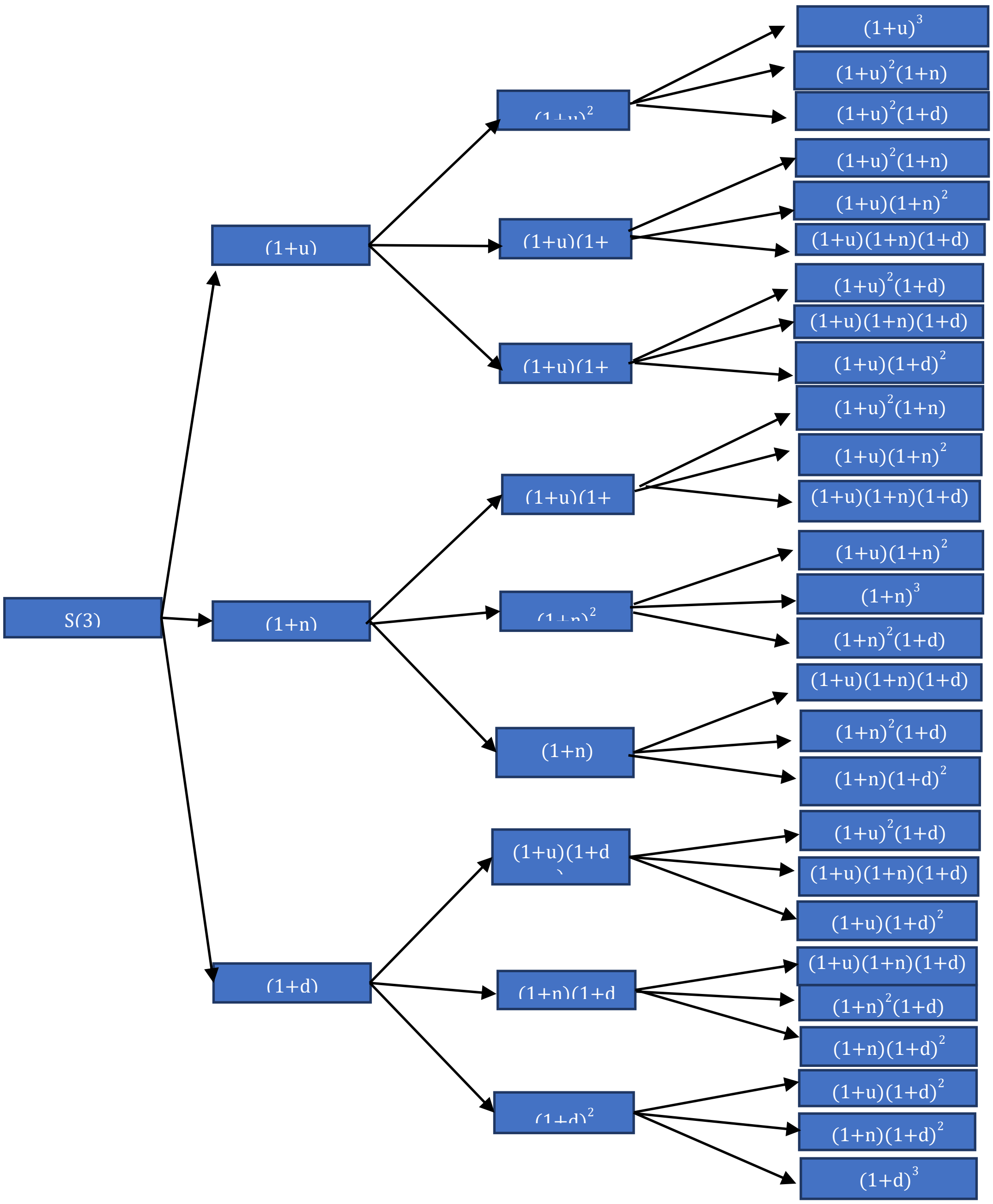

Figure 4: 2nd and 3rd step of price movement

Similarly, we will get the values of S(4), S(5),…… so on.

The values of S(n) along with the corresponding probabilities can be found for any n by extending in the above way.

The general formula can be written:

$S(n)=S(0)(1+u)^k(1+n)^l(1+d)^{n-k-l}$ with probability $\binom{n}{k}\binom{n-k}{l}{p_u}^k {p_m}^l {p_d}^{n-k-l}$

Where k=0,1,2,….,n and l=0,1,2,…..,n.

4. Experiment with Some Problems

4.1Problem

For comparison between binomial and trinomial stock pricing, assume stock price initially, S(0) = 50 = X ;Volatility σ = 0.25; interest rate, r = 0.1,Time period, T=1 years and time step N=3 (Hull, 2018).

We consider a non-dividend paying stock for the purpose of the illustration i.e. q=0 pricing using trinomial tree with their probability. For trinomial:$u = e^{\sigma\sqrt{2\Delta t}}; d = e^{-\sigma\sqrt{2\Delta t}} = \frac{1}{u}; m = 1$

And the corresponding probabilities are

$$p_u = \left(\frac{e^{\frac{(r-q)\Delta t}{2}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}}{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}}\right)^2$$

$$p_d = \left(\frac{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{\frac{(r-q)\Delta t}{2}}}{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}}\right)^2$$

$$p_m = 1 -(p_u + p_d)$$

Calculating we get, u=1.2265, d = 1/u=0.8153, m=1;

p_u=0.3099 , p_d=0.08054, p_m=0.6095

then Using equation (1),S (3)

$$= \begin{cases} 27.067 \text{ with probability } 0.005, \\ 40.71 \text{ with probability } 0.006, \\ 61.25 \text{ with probability } 0.023, \\ 92.14 \text{ with probability } 0.029, \\ 33.21 \text{ with probability } 0.011, \\ 40.75 \text{ with probability } 0.089, \\ 50 \text{ with probability } 0.226, \\ 49.96 \text{ with probability } 0.091, \\ 61.30 \text{ with probability } 0.345, \\ 75.15 \text{ with probability } 0.175, \\ , \end{cases} \quad \ldots\ldots\ldots\ldots(2)$$

To compare we have to first calculate the rising and falling pricing movement for the binomial tree model and also calculate the probability of these stock price movement scenario.

We know the formula for Binomial model from the original (Cox et. al. 1979) method

Binomial:

$$u = e^{\sigma\sqrt{\Delta t}};\ d = e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u}$$

And probabilities

$$p = \left(\frac{e^{(r-q)\Delta t} - d}{u - d}\right)^2$$

Using previous value we can estimate for binomial model u=1.1553,d=0.8656 and p=0.58;

Here for Binomial tree we have from (Capinski and Zastawniak, 2003)

$S(n) = S(0)(u)^i(d)^{n-i}$ with probability $\binom{n}{i}p^i(1-p)^{n-i}$

Then,S (3)

$$= \begin{cases} 32.42 \text{ with probability } 0.074 \\ 58.18 \text{ with probability } 0.3069 \\ 104.38 \text{ with probability } 0.4238 \\ 187.83 \text{ with probability } 0.1950 \end{cases} \quad \ldots\ldots\ldots(3)$$

In Trinomial model,

The stock values of S(n) at time n is a discrete random variable with 2n + 1 different values(Capinski et. al. 2003) which we can see from (2), 40.75 with probability 0.089 and 40.71 with probability 0.006 Finally added the probability we have 40.7 with probability 0.095. In a similar manner, S(3) has 2*3+1=7 different increasing values respectively and their probability as given below:

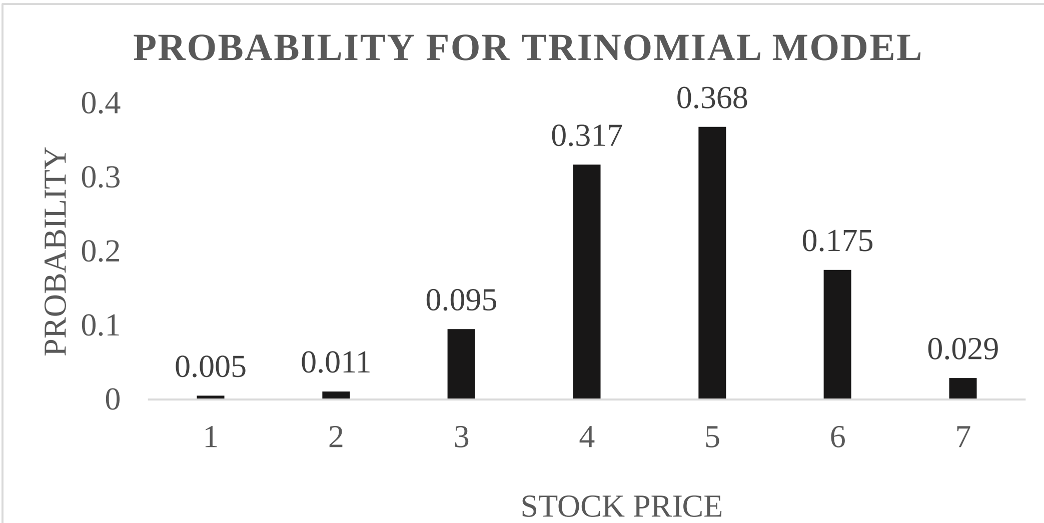

Figure 5: Probability of Trinomial Stock price

The stock values of S(n) at time n is a discrete random variable with n + 1 different values (Capinski et. al. 2003). Using (3) also we have, S(3) has 3+1=4 different increasing values respectively and their probability as given below:

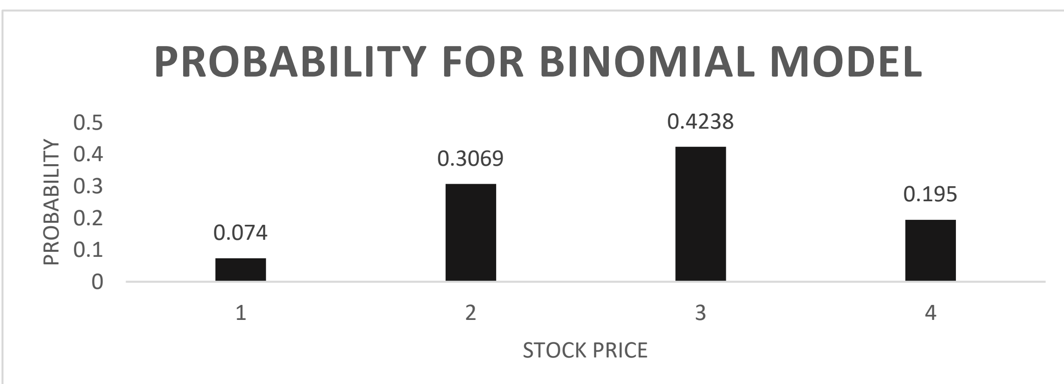

Figure 6: Probability of Binomial Stock price

We can see above the graph comparison, trinomial model gives more price movements with probability which is helpful for traders.

4.2European and American Option Pricing

Now we go for American and European option pricing, for this first of all we have to know the basic difference between them. European options keep within bounds the holder to exercise only at the expiration date, while American options offer the flexibility to exercise at any point before the expiration date. This key difference can influence the value and behavior of the options in various market conditions and trading strategies.

Problem: Consider initial stock price S (0)=50 = X , and annual standard deviation is 27%andreturn rate is 10%.Mature date of the stock priceis1yearandCalculate stock price for American and European option price.(Hull, 2018)

We have the formula for 'u', 'd' and 'm' respectively :$u = e^{\lambda\sigma\sqrt{\Delta t}}; d = e^{-\lambda\sigma\sqrt{\Delta t}} = \frac{1}{u}; m = 1$

Boyle used multiple value for λ,but Hull used $\lambda = \sqrt{3}$,then we get u=1.3099; m=1; d=0.7634 then our three step tree:

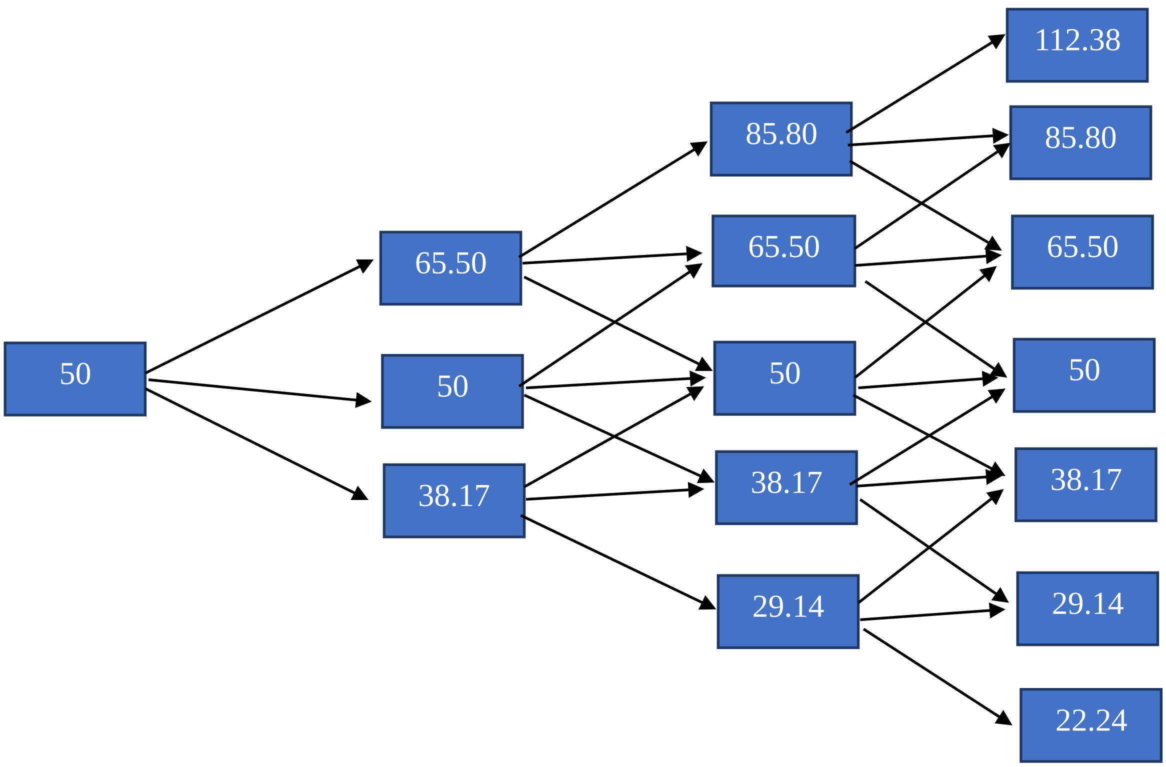

Figure 7: Numerical Example of Trinomial Stock Price Lattice

For upward price movement probability P_1 , remain neutral P_2 and for downward movement probability P_3 .We used the solutions for P_1 , P_2 , P_3 as

$$p_1 = \frac{1}{6} + \sqrt{\left(\frac{h}{12\,\sigma^2}\right)}\,(r - q - \frac{\sigma^2}{2});$$

$$p_2 = \frac{2}{3};$$

$$p_3 = \frac{1}{6} - \sqrt{\left(\frac{h}{12\,\sigma^2}\right)}\,(r - q - \frac{\sigma^2}{2});$$

Calculating for this problem we have, p_1 = 0.2059 ,p_2 =0.67,p_3 =0.1274

call option pay-off formula C = Max(0, S – X).

The payoffs are represented by cf1, cf2, and cf3 cf1 = Max(0,112.38-50) = 62.38 , similarly we get cf2 , cf3.

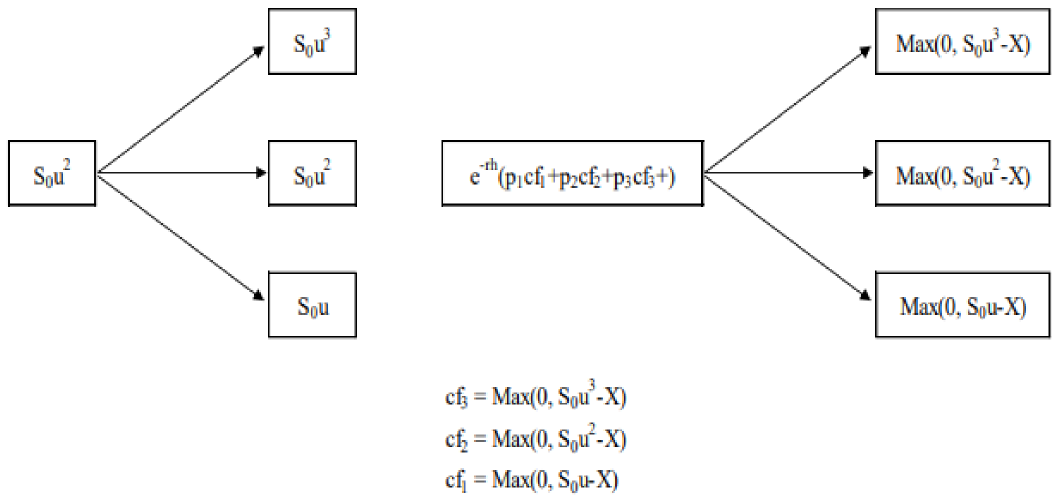

Figure 8: One step (Kuang et. al. 2022) **back valuation**

Now we calculating by back valuation we get,

$$call\ option\ , c = (62.38*\ 0.2059 + 35.80*\ 0.667 + 15.50*\ 0.1274))\ e^{(\ -(\ 1/3\ )*0.04)} = 38.29$$

Continuing this process ,

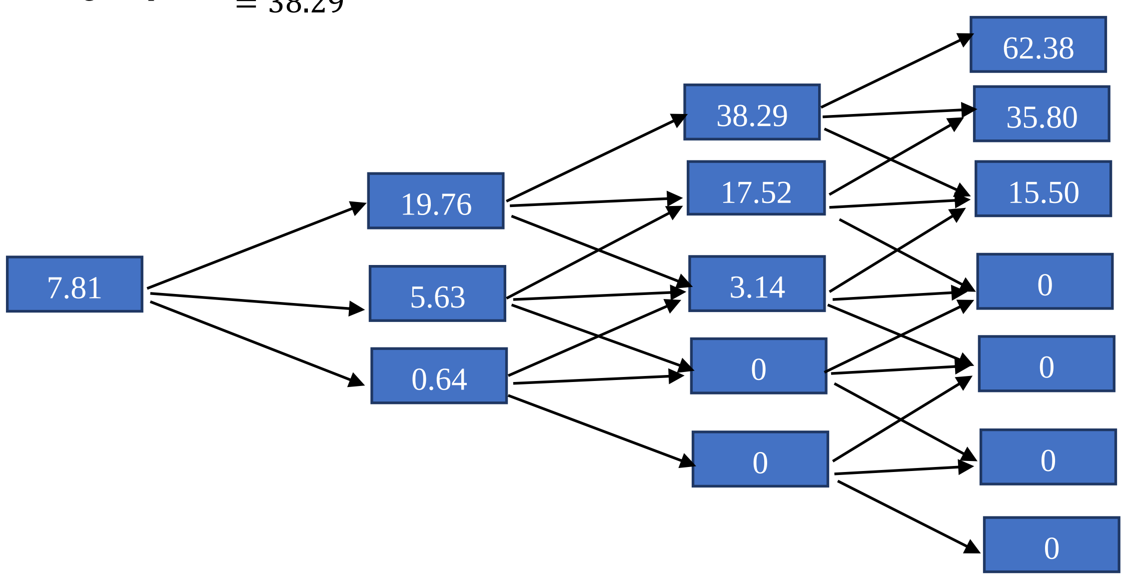

Figure 9: European Call Option Price

put option pay-off formula : P = Max(0, X – S)

cf1 = Max(0,50-38.17) = 11.83 , similarly we get cf2 , cf3.

$p = e^{(-(1/3)*0.04)}(11.83* 0.2059 + 20.86* 0.667 + 27.76* 0.1274) = 19.62$

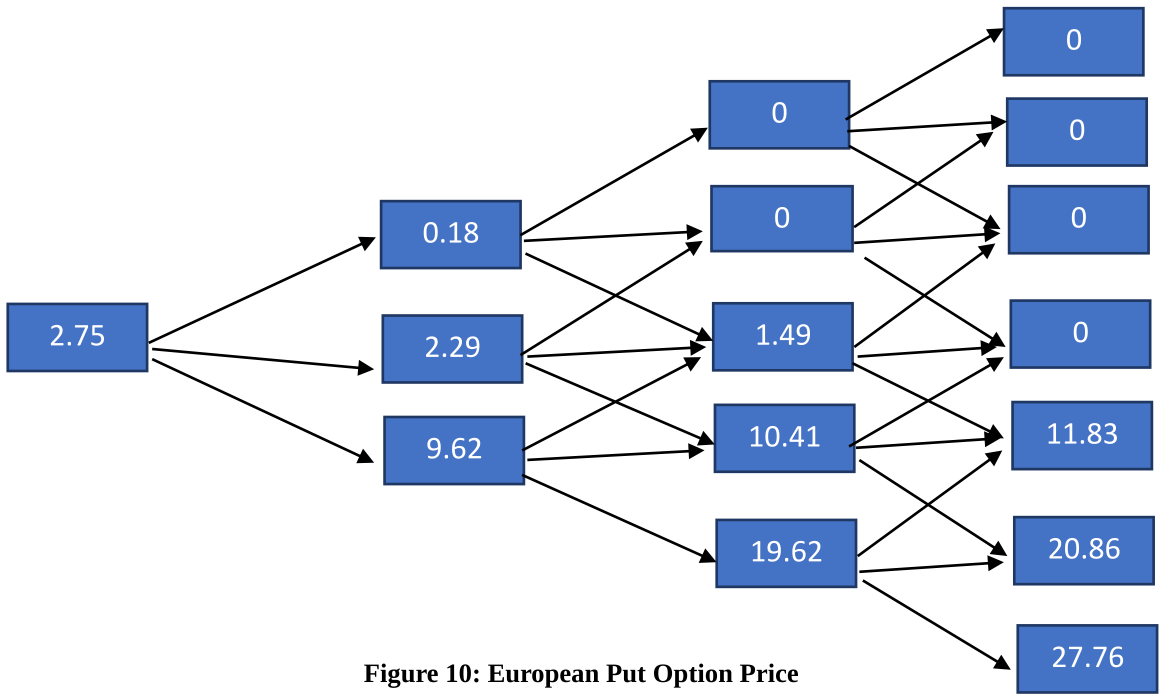

Figure 10: European Put Option Price

Now, for American option pricing we have the opportunity to get the highest value. In this problem, 50 is the strike price, while 85.80 is the stock price, at the respective node. The calculations at each of the nodes have to be adjusted accordingly.

c=Max $(85.80 - 50, e^{(-(1/3)*0.04)}(62.38 * 0.2059 + 35.80 * 0.667 + 15.50 * 0.1274))$=38.29

Each node's value is determined by looking at the values of three nodes that come after it, as well as any pay-offs from ending the process early. Repeating this back valuation system, the price of the American call can be calculated at time = 0.

In the example considered American call option price is equal to the European call option price under certain conditions, typically when there are no dividends paid on the underlying asset and no opportunity for early exercise. Similarly, the American put option price can be determined by adjusting the European put option price. Then we have,

p=Max $(50 - 29.14, e^{(-(1/3)*0.04)}(11.83 * 0.2059 + 20.86 * 0.667 + 27.76 * 0.1274) = 20.86$ Then for each node our tree arrived as:

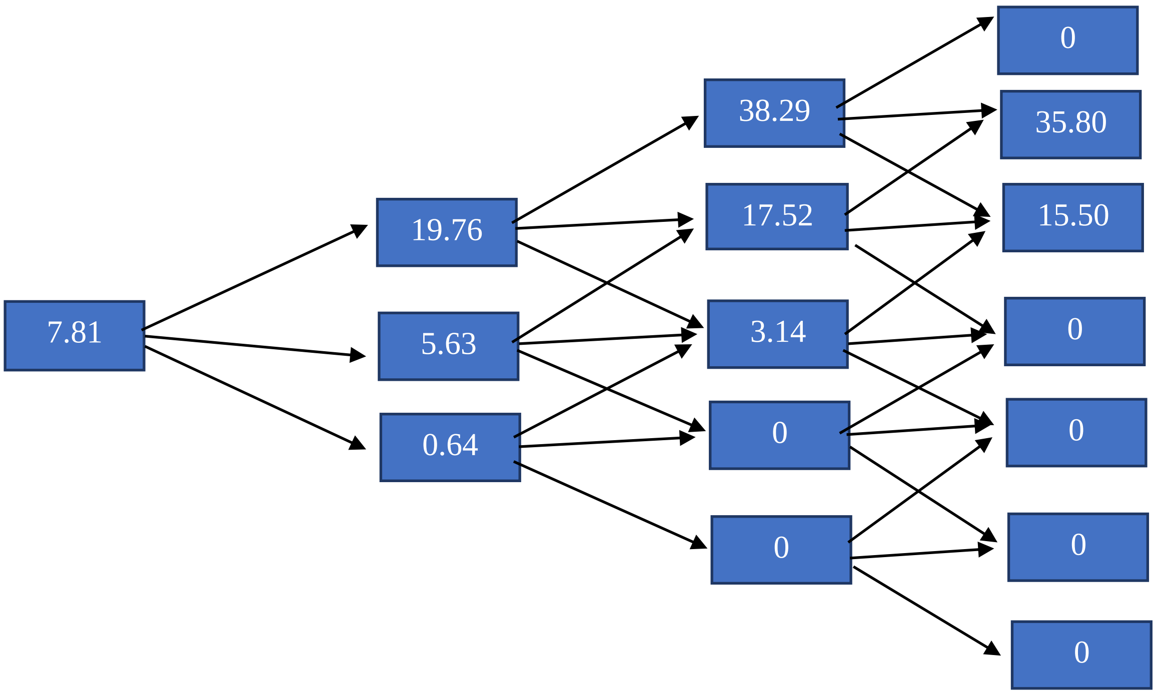

Figure 11: American Call Option

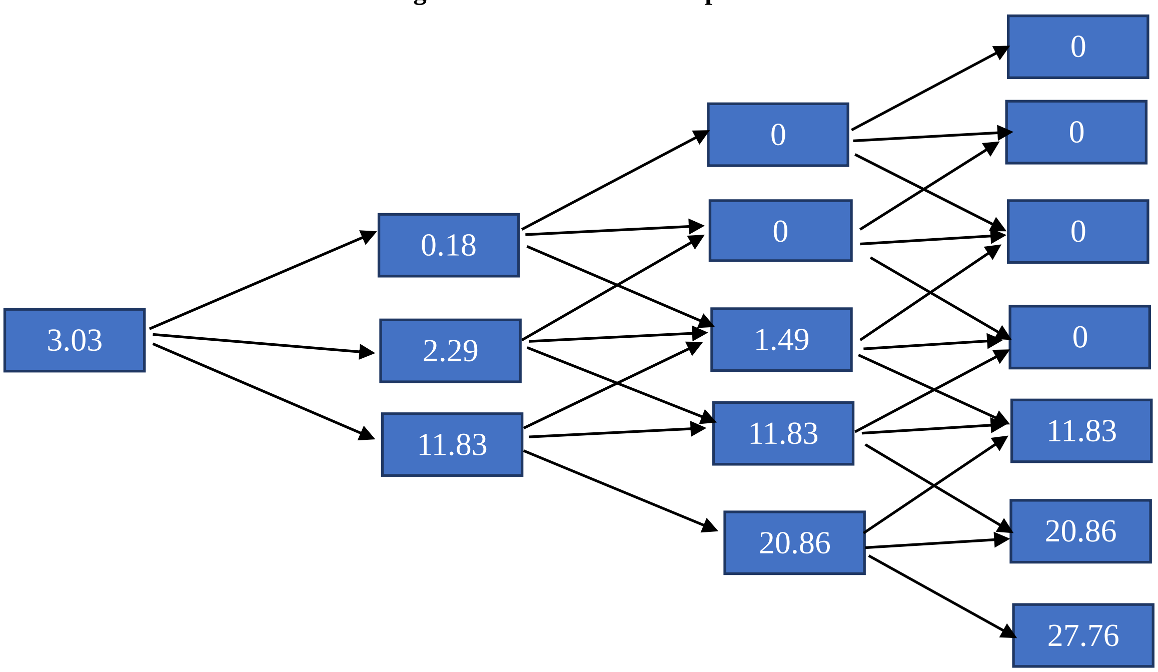

Figure 12: American Put Option

In the example being considered, we see that the American put option is priced higher than the European put option.

Conclusion

In this paper, we focused on basic concepts of trinomial tree model and we showed the difference between BM and TM with probability. Then, we apply the Trinomial model for American and European option pricing. American options offer more flexibility but come with higher premiums, while European options are less flexible (limited) but have lower premiums. Traders can choose any of them depending on their trading strategy, market demands, and risk tolerance. The future scope for trinomial tree models is vast and dynamic, driven by advancements in financial engineering, computational methods, and market dynamics. Continued research and innovation in this field are essential for improving the accuracy, efficiency, and applicability of trinomial tree models in financial markets and risk management practices.

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