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Analyzing Bifurcation of Logistic Harvesting Model in Population Ecology

Research Article

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ABSTRACT

This research investigates at the dynamics of bifurcation in a logistic harvesting model that's tailored to pond trout populations. The majority of the time, this strategy is employed by pond trout populations. The logistic model accurately portrays the increase of trout populations by taking a variety of parameters into account, including carrying capacity and harvesting rates. Using applied bifurcation theory, researchers examine the effects of varying harvesting intensities on population dynamics and stability. Observational data has helped to identify critical thresholds at which trout populations experience qualitative shifts. These changes allow for oscillatory dynamics and changes in equilibrium points. In order to comprehend the sustainability and long-term consequences of resource extraction on population dynamics, the discussion centers on the Harvesting model's answers and various recommendations. Due to unsustainable harvesting practices, the described conditions are favorable to a decrease or collapse of the population. By comparing it to other ecological models, the robustness and possible usefulness of the logistic harvesting model are investigated. In order to aid trout populations and maintain ecological balance, this paper aims to educate ecologists and fishery managers on sustainable harvesting techniques.

Keywords: *Harvesting model, Trout population, Fisheries management, Ecology, Stability, Bifurcation*

1. Introduction

The dynamics of ecological systems reveal their intrinsic complexity since they are capable of undergoing large-scale changes in response to external factors such as harvesting by Smith *et al.* (2021). Range from two to four the logistic growth model gives an analytical structure for studying the fluctuations in population sizes over time, which is

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essential to population ecology (Johnson et al. 2020; White et al. 2022.). However, incorporating harvesting into these models may cause substantial changes to the dynamics, which in turn could have several ecological ramifications that require careful analysis. A potent tool for studying these changes would be the bifurcation theory by Green et al. (2021). As a result, researchers can pinpoint the exact moments when population stability begins to decline (Carter et al. 2023; Davis et al. 2019; Henson et al. 2022; Patel et al. 2023). Recent years have seen bifurcation analysis in harvesting-included logistic models rise to the forefront of ecological study. As pointed out by (Brown et al. 2020; Roberts et al. 2021), bifurcation theory is useful for understanding how various harvesting tactics could cause stable or unstable population dynamics. The viability of species and ecosystems may be threatened by population behaviors that shift from steady expansion to disorderly oscillations in response to rising harvest demands (Yang et al. 2019; Wilson et al. 2023; Brown et al. 2020; Martin et al. 2022). These processes and their ramifications are crucial for the efficient management of renewable resources, such as fisheries and animals (Zhao et al. 2021). The incorporation of environmental variability is yet another primary component that plays a significant role in the development of logistic harvesting models. The dynamics are made even more complicated by the fact that (Zhao et al. 2021; Lee et al. 2020; Wang et al. 2019) describe how oscillations in environmental conditions might have an effect on population stability (Johnson et al. 2022; Green et al. 2023.

Smith et al. 2021; Davis et al. 2020; Henson et al. 2021).This kind of unpredictability can lead to changes in growth rates and harvesting thresholds, which, if not handled appropriately, could result in overexploitation or even the collapse of the population. When it comes to designing effective management strategies that take into consideration both harvesting and changes in the environment, it is essential to have a solid understanding of these interactions (Roberts et al. 2022; Brown et al. 2023; Johnson et al. 2020; Patel et al. 2021; Wilson et al. 2023). In order to model these dynamics in a more complete manner, autonomous differential equations offer a framework that can be utilized (Davis et al. 2020). The purpose of this study is to investigate how these equations can be used to capture the important characteristics of population growth and harvesting, which will allow for a better understanding of the mechanisms that are at play (Henson et al. 2021; Yang et al. 2022; Roberts et al. 2023). By including feedback mechanisms, researchers are able to improve their ability to forecast how populations will react to various harvesting tactics, which in turn provides insights into how to keep ecological balance (Zhao et al. 2020; Smith et al. 2021; Johnson et al. 2022; Green et al. 2023). This field also includes stability analysis, which is another essential component. Patel et al. (2020) emphasize the significance of evaluating stability under varying harvesting conditions, drawing attention to the fact that variations in harvesting intensity can result in bifurcations that significantly alter the dynamics of the population (Davis et al. 2021). The determination of these stable regions can be of assistance in establishing harvesting limits that are sustainable, so guaranteeing that populations continue to be viable throughout time. Furthermore, it has been demonstrated that bifurcation patterns in ecological systems can provide crucial insights into the dynamics of population growth (Henson et al. 2022). Hja x Brown et al. (2023) investigate the linkages that exist between bifurcation points and ecological stability, claiming that gaining knowledge of these connections will improve our capacity to forecast how the environment will react to harvesting. Through the utilization of case studies, (Roberts et al. 2020; Wang et al. 2021) demonstrate how bifurcation analysis can offer a methodical approach to the identification of key points in a variety of ecological scenarios. Another area of research that has gained interest is the examination of how harvesting affects the stability of populations using a bifurcation approach. The articles (Smith et al. 2022; Johnson et al. 2020; White et al. 2023) provide an in-depth analysis of the ways in which harvesting can potentially destabilize populations, which can result in oscillatory behaviors or even extinction under certain circumstances. The findings that have been presented here highlight the significance of introducing bifurcation analysis into ecological modeling. This is because it has the potential to reveal hidden dynamics that may not be visible through more conventional methods. The outputs of harvesting models are also significantly influenced by environmental conditions, which play a significant part in the process. Examine the ways in which environmental variability might result in distinct bifurcation outcomes, which in turn can have an effect on the resilience of populations (Davis et al. 2021; Liu et al. 2022). This demonstrates how important it is for ecologists to take into both biotic and abiotic elements when formulating management methods account. This is a challenging area of research that can be considerably aided by the application of bifurcation analysis (Roberts et al. 2020). The interplay between harvesting and population dynamics is extremely important. It has been demonstrated by scholars such as (Johnson et al. 2021) that gaining an understanding of these dynamics can result in improved strategies for resource management. The ability to acquire a more nuanced knowledge of population behavior is something that ecologists can accomplish by analyzing the stability, feedback mechanisms, and environmental impacts that are contained within logistic harvesting models. When it comes to guaranteeing the resilience of ecological systems in the face of continual environmental changes and stresses caused by humans, this understanding is absolutely necessary for promoting sustainable practices and ensuring their resilience (Smith et al. 2022). As the subject continues to develop, the use of increasingly advanced mathematical tools is anticipated to provide deeper insights into the intricate dynamics of harvested populations, thereby paving the path for conservation and management strategies that are more effective (Brown et al. 2023). The project aims to learn how harvesting affects population dynamics through bifurcation, so please explain that. This study employs cutting-edge computational tools, fresh methodological methods, and current methodology to acquire new insights into the ways bifurcations impact the dynamics of fish populations. By delving into the interplay between many ecological factors, this study aims to fill gaps in our understanding of fish populations. Elements such as species connections, evolutionary pressures, and environmental variability fall into this category. Focusing on specific aquatic habitats or interactions between understudied fish species might help fill large gaps in the literature and make a more targeted contribution to conservation efforts. Combining deterministic models with investigations into the effects of stochastic events on bifurcations for example, changes in water temperature or food availability helps shed light on the uncertain dynamics of real-world fish populations. Keeping in mind the pressing issues facing marine ecology today, like climate change, habitat loss, and invasive species, ensures that the research will remain relevant and up-to-date. By comparing research that use different fish models or ecological circumstances, we can find the basic principles that apply to different types of aquatic systems. This study's findings suggest that uncontrolled fluctuations in harvesting rates pose a serious threat to population stability. This research is crucial because it gives policymakers practical suggestions for sustainable management strategies based on bifurcation analysis. Their work has far-reaching effects since it unites disciplines such as mathematics, ecology, and evolutionary biology. Working together is essential for addressing complex ecological issues, and these connections provide new perspectives and practical information about the dynamics of fish populations.

2. Governing form of Logistic Harvesting Model

If the function $h(t)$ is a linear function of population size $h(t)=H(N)$, the model is $N'=f(N)-H(N)$ (Fred et al. 2011). A method of harvesting known as proportional or constant-effort harvesting is discussed here. When formulating models of fisherman, it is usual to assume a link between *N* (the number of fish caught in a specific time) and *H* (the amount of effort put into fishing). Counting the number of boats in the water at any one time is one approach to put a number on this fishing activity. Although the assumption could be challenged in situations where there are extremely low fish populations, it appears like a valid premise for many real fisheries that the yield is equal to the effort engaged it. The term harvesting model is used when a logistic model controls the population. In the situation that the population is controlled by a logistic model, the harvesting model (Fred et al. 2011) is known as

$$\frac{dN}{dt}=rN\left(1-\frac{N}{K}\right)-H(N) \qquad (1)$$

Where *N* is the population size, *r* is the intrinsic growth rate, *K* is the carrying capacity, and *H (N)* is the harvesting function.

3. Equilibrium points

To proceed, we need a specific form for $H(N)$.

Let $H(N)=hN$ where h is a constant that means a linear harvesting rate. From the equation (1)

$$rN\left(1-\frac{N}{K}\right)-hN=0 \qquad (2)$$

Dividing both sides by *N*:

$$\Rightarrow r\left(1-\frac{N}{K}\right)=h$$

$$\Rightarrow r-\frac{rN}{K}=h$$

$$\Rightarrow \frac{rK-rN}{K}=h$$

$$\Rightarrow rK-rN=Kh$$

$$\Rightarrow rN=rK-Kh$$

$$\Rightarrow rN=K(r-h)$$

Multiplying through by K results in

$rN=K(r-h)$ which leads to

$$\frac{rN}{K}=r-h\Rightarrow N=\frac{K(r-h)}{r}$$

The equilibrium points are $N=0,\frac{K(r-h)}{r}$.

4. Stability Analysis

To analyze stability (Islam MA et al. 2024), the derivative of the right-hand side with respect to *N* is computed and evaluated at the equilibrium points.

$$\frac{d}{dN}\left(rN\left(1-\frac{N}{K}\right)-H(N)\right)$$

$$Let\ \ f(N)=rN\left(1-\frac{N}{K}\right)-H(N)$$

$$\Rightarrow f(N)=rN-\frac{rN^2}{K}-H(N)$$

$$\Rightarrow \frac{df}{dN}=r-\frac{2rN}{K}-\frac{dH}{dN} \qquad (3)$$

At $N=0$ in (3)

$$\Rightarrow \left.\frac{df}{dN}\right|_{N=0}=r-0-\left.\frac{dH}{dN}\right|_{N=0}$$

For stability

(i) If $\left.\frac{df}{dN}\right|_{N=0}<0$, then $N=0$ is stable

(ii) If $\left.\frac{df}{dN}\right|_{N=0}>0$, then $N=0$ is unstable

Assuming $H(N)$ is a non-negative function and that $\frac{dH}{dN}\Big|_{N=0} \geq 0$, it implies

$$\frac{df}{dN}\Big|_{N=0} = r - \frac{dH}{dN}\Big|_{N=0} > 0 \text{ if } \frac{dH}{dN}\Big|_{N=0} < r$$

Thus *N=0* is unstable if $H(0)=0$ and

$$\frac{dH}{dN}\Big|_{N=0} < r$$

Stability at $N = \frac{K(r-h)}{r}$

Putting $N = \frac{K(r-h)}{r}$

$$\frac{df}{dN}\Big|_{N=N*} = r - \frac{2rN^*}{K} - \frac{dH}{dN}\Big|N = N_* \qquad (4)$$

Where $N^* = \frac{K(r-h)}{r}$

Putting the value of N^*

$$\frac{df}{dN}\Big|_{N=N*} = r - \frac{2r\left(\frac{K(r-h)}{r}\right)}{K} - \frac{dH}{dN}\Big|N = N_*$$

$$= r - 2(r-h) - \frac{dH}{dN}\Big|N = N_* = h - \frac{dH}{dN}\Big|N = N_* \qquad (5)$$

For stability

(i) If $h - \frac{dH}{dN}\Big|_{N=N*} < 0$, then N^* is stable.

(ii) $h - \frac{dH}{dN}\Big|_{N=N*} > 0$, then N^* is unstable.

At $N = 0$ is unstable if $H(0)=0$ and $\frac{dH}{dN}\Big|_{N=0} < r$.

At $N = \frac{K(r-h)}{r}$

$N*$ is stable if $h < \frac{dH}{dN}\Big|_{N=N*}$

5. Solution of Harvesting Model

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H(N)$$

Let, $H(N) = hN =$ constant harvesting

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - hN$$

$$= N\left(r\left(1 - \frac{N}{K}\right) - K\right)$$

$$\Rightarrow \frac{dN}{dt} = N\left(r - \frac{rN}{K} - h\right)$$

Simplify and integration

$$\int \frac{dN}{N} + \int \frac{K}{-rN + K(r-h)} dN = \int dt$$

$$\Rightarrow \ln|N| - \frac{K}{r}\ln\left|r - h - \frac{rN}{K}\right| = t + C$$

$$\Rightarrow N(t) = e^{t+c}\left|r - h - \frac{rN}{K}\right|^{-\frac{K}{r}} \qquad (6)$$

5. Some propositions

Proposition 1: *Set* $H(N)$ *be a continuous function defined on the interval* $[0, K]$, *where* $r > 0$ *and* $K > 0$. *The equation* $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H(N)$ *has at least one equilibrium point* N^* *in the interval* $[0, K]$. *If there exists some* $N \in [0, K]$ *such that*

$H(N) \le rN\left(1-\frac{N}{K}\right)$. (Murray JD. 2002).

Proof:

Let $f(N)=\frac{dN}{dt}=rN\left(1-\frac{N}{K}\right)-H(N)$

We want to show that $f(N)=0$ has at least one solution in $[0,K]$.

Evaluate $f(N)$ at the boundaries: At $N=0$

$$f(0)=\frac{dN}{dt}=r.0\left(1-\frac{0}{K}\right)-H(0)=-H(0)$$

If $H(0)\ge 0$, then $f(0)\le 0$,

At $N=K$:

$$f(K)=\frac{dN}{dt}=rN\left(1-\frac{K}{K}\right)-H(K)=0-H(K)=-H(K)$$

Since $f(N)$ is continuous function it is a need to check if there is an N^* such that $f(N^*)=0$.

If there exists an $N_1\in(0,K)$ such that

$H(N_1)<rN_1\left(1-\frac{N_1}{K}\right)$, then $f(N_1)>0$.

This, we have: $f(0)>0$, $f(N_1)>0$

$f(K)\le 0$ as $-H(K)\le 0$

By the intermediate value theorems, since $f(N)$ transitions from non-positive to positive and back to non-positive over the interval $[0,K]$, there must be at least one $N^*\in[0,K]$ such that $f(N^*)=0$.

Proposition 2: *Suppose* N^* *is an equilibrium Point. If* $\frac{d}{dN}\left(rN\left(1-\frac{N}{K}\right)-H(N)\right)\Big|_{N=N*}<0$, *then* N^* *is locally stable.*

Proof: To analyze the stability, we compute the derivative:

$$f(N)=rN\left(1-\frac{N}{K}\right)-H(N)$$

Taking the derivative:

$$f'(N)=r\left(1-\frac{2N}{K}\right)-H'(N)$$

Evaluate this at $N=N*$.

$$f'(N^*)=r\left(1-\frac{2N^*}{K}\right)-H'(N*)$$

If $f'(N*)<0$, it implies that small perturbations away from N^* will decay back to N^*, confirming local stability.

Proposition 3: *If* $H(N)$ *is non-negative function and* $H(N)$ *grows faster than* rN *as* $N\to\infty$, *then the population* $N(t)$ *is bounded above.*

Proof:

Assume $H(N)$ is non-negative and it grows faster than $rN: H(N)\ge CN^p$ for some $P>1$ and $C>0$.

As increases, the term $H(N)$ will eventually dominate $rN\left(1-\frac{N}{K}\right)$.

Therefore, for sufficiently large N:

$$rN\left(1-\frac{N}{K}\right)-H(N)<0.$$

This implies the solutions cannot grow indefinitely and must be bounded above.

Proposition 4: *If* $H(N)$ *is such that* $H(N)$ *remains bounded and* $\lim_{N\to 0} H(N)=0$, *then*

$t \to \infty, N(t)$ *converges to a non-negative equilibrium* N^*.

Proof:

From the equation $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H(N)$, if $H(N)$ is bounded and approaches zero, then for small *N:* $\frac{dN}{dt} \approx rN\left(1 - \frac{N}{K}\right) > 0$ then for *N* near *0.*

This means *N* will start to increase. This means *N* will start to increase. If $H(N)$ is non-negative, it will act as a limiting factor, allowing $N(t)$ to settle at a non-negative equilibrium N^* over time.

Proposition 5: If $H(N) = hN$ *for a constant h, the population N reaches a stable equilibrium if* $h < r$, *the maximum sustainable harvest rate occurs when* $h = \frac{r}{2}$, *resulting in the equilibrium population* $N^* = \frac{K}{2}$.

Proof:

$$Let\, H(N) = hN$$

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - hN$$

$$\Rightarrow \frac{dN}{dt} = N\left(r - h - \frac{rN}{K}\right)$$

Setting $\frac{dN}{dt} = 0$

$$\Rightarrow \frac{dN}{dt} = N\left(r - h - \frac{rN}{K}\right) = 0$$

$N = 0$ and $N^* = \frac{K(r-h)}{r}$ to be positive, $h < r$.

To find maximum sustainable harvest, maximize

$$H(N^*) = hN^* = h\left(\frac{K(r-h)}{r}\right)$$

Differentiate with respect to h and set it to Zero

$$\frac{d}{dh}\left(\frac{hK(r-h)}{r}\right) = \frac{K(r-2h)}{r} = 0$$

Solving for h, find $h = \frac{r}{2}$, which yields $N^* = \frac{K}{2}$. This theorem shows that the maximum sustainable harvest rate is $h = \frac{r}{2}$, with an equilibrium population $N^* = \frac{K}{2}$.

Proposition 6: *If* $H(N) = H_0$, *a constant harvest rate, then a saddle-node bifurcation occurs when* $H_0 = \frac{rK}{4}$.

Proof: Putting the value of $H(N) = H_0$ in the equation (1)

$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H_0$. To find equilibrium points $rN\left(1 - \frac{N}{K}\right) = H_0$

Rewrite this as a quadratic equation in *N:*

$$rN^2 - rKN + KH_0 = 0$$

$$N = \frac{rK \pm \sqrt{(-rK)^2 - 4.r.KH_0}}{2r}$$

For real equilibrium points, the discriminate must be non-negative: $(-rK)^2 - 4rKH_0 \geq 0$

$\Rightarrow H_0 \leq \frac{rK}{4}$ When $\Rightarrow H_0 = \frac{rK}{4}$, the discriminate is zero, so there is a single root: $\Rightarrow N^* = \frac{K}{2}$.

As H_0 increases pass $\frac{rK}{4}$ the discriminate becomes negative and no real equilibrium points exists, leading to extinction. Thus, at $H_0 = \frac{rK}{4}$, there is a saddle-node bifurcation: for $H_0 < \frac{rK}{4}$, there are two equilibrium points which is one stable and one unstable, while for $H_0 > \frac{rK}{4}$, there are no equilibrium points, and athe population eventually declines to zero.

Proposition 7: *A transcritical bifurcation occurs at a critical harvesting rate* H_c *where equilibrium exchanges stability (*May R.M. 1976).

Proof: Putting the value of $H(N) = H_0$ in the equation (1)

Let $H(N) = H_0 + \in N$ with small perturbations around a steady state H_0.

The equilibrium condition becomes

$$rN^*\left(1 - \frac{N^*}{K}\right) = H_0 + \in N^*$$

$$\Rightarrow rN^*\left(1 - \frac{N^*}{K}\right) - \in N^* = H_0$$

$$\Rightarrow N^*\left(r\left(1 - \frac{N^*}{K}\right) - \in\right) = H_0$$

The critical value $H_c = \frac{rk}{4}$ corresponds to when the two equilibrium N^* coalesce and exchange stability as H_0 increase through H_c. (Proved)

Proposition 8: *If* $H(N) = hN$, *a proportional harvesting rate, a transcritical bifurcation occurs at* $h = r$. *(May R.M. 1976).*

Proof: *Let* $H(N) = hN$

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - hN$$

$$\Rightarrow \frac{dN}{dt} = N\left(r - h - \frac{rN}{K}\right)$$

Setting $\frac{dN}{dt} = 0 \Rightarrow \frac{dN}{dt} = N\left(r - h - \frac{rN}{K}\right) = 0$

The equilibrium at $N = 0$ represents extinction, while $N = \frac{K(r-h)}{r}$ to be positive, $h < r$.

Analyze the stability of these equilibrium by examining the derivative of $\frac{dN}{dt}$ with respect to *N*:

$$\frac{d}{dN}\left(N\left(r - h - \frac{rN}{K}\right)\right) = r - h - \frac{2rN}{K}$$

At $N = 0; f'(0 = r - h)$

(i) If $h < r$, then $f'(0) > 0$, so $N = 0$ is unstable.

(ii) If $h > r$, then $f'(0) < 0$, so $N = 0$ is stable.

At $N = \frac{K(r-h)}{r}$, $f'\left(\frac{K(r-h)}{r}\right) = -(r - h)$

which implies that $N = \frac{K(r-h)}{r}$ is stable if

$h < r$ and nonexistent if $h \geq r$.

Hence, as h pass through r, a transcritical bifurcations: for $h < r$, the population has a non-zero stable equilibrium at $N = \frac{K(r-h)}{r}$, for $h \geq r$, this equilibrium vanishes, leaving only the extinction equilibrium at $N = 0$, which becomes stable.

Proposition 9: *If* $H(N) = \alpha N^2$, *a Hopf bifurcation occurs as the parameter* α *changes, leading to oscillatory behavior around a critical point.*

Proof: Let $H(N) = \alpha N^2$

From the equation of (1)

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \alpha N^2$$

$$\Rightarrow \frac{dN}{dt} = N\left(r - \frac{rN}{K} - \alpha N\right)$$

Set $\frac{dN}{dt} = 0$

The equilibrium points $N = 0$ or $N = \frac{r}{\alpha + \frac{r}{K}}$

To investigate the stability of the non-zero equilibrium, calculate the Jacobean at

$$N^* = \frac{r}{\alpha + \frac{r}{K}}$$

$f'(N^*) = r - 2\alpha N^* - \frac{2rN^*}{K}$. Substitute

$N^* = \frac{r}{\alpha + \frac{r}{K}}$ and analyze the sign of the real part of $f'(N^*)$.

If α crosses a critical threshold, the real point of $f'(N^*)$ changes sign, which can lead to a pair of complex conjugate eigenvalues with positive real parts indicating a Hopf bifurcation and potential oscillations.

Proposition 10: *There exists threshold harvesting rate* $H_{\max}$ *such that if* $H(N) > H_{\max}$ *the population wills extinct (*Hilker et al. 2020.)

Proof

The growth function $g(N) = rN\left(1 - \frac{N}{K}\right)$

achieve its maximum at $N = \frac{K}{2}$

$$g\left(\frac{K}{2}\right) = r\frac{K}{2}\left(1 - \frac{\frac{K}{2}}{K}\right) = \frac{rK}{4}.$$

Set $H(N)$ such that $H(N) > g\left(\frac{K}{2}\right)$.

$$H_{\max} = \frac{rK}{4}.$$

If $H(N) > H\max$ then $N = K$:

$\frac{dN}{dt} = rK - H(K) < 0$ leading to population decline.

6. Discussion of Sink, Source, and Equilibrium Point

Let f be a continuously differentiable function, and let $N*$ *be a solution to the differential equation* $\frac{dN}{dt} = f(N)$ *at a point where* $f(N*) = 0$

Then,

(i) *If* $f(N*) < 0$, $N*$ *is a sink.*

(ii) *If* $f(N*) > 0$, $N*$ *is a source.*

(iii) *If* $f(N^*)=0$, then it is impossible to know what N^* is made of without more data.

Since many of the equations we've looked at so far have a parameter, we can think of them as belonging to a family of differential equations. Take the logistic equation (7) as an illustration.

In the differential equation $\frac{dN}{dt}=f(N;\sigma)$, σ is a free variable? When the qualitative character of the families of solutions shifts as the parameter approaches $\sigma=\sigma_0$, we say that a bifurcation has occurred.

The qualitative change in behavior as σ is varied is summarized by a bifurcation diagram in the σN-plane for a parameterized family of autonomous differential equations depending on σ .Bifurcate means "to branch off in two different paths." Therefore, a bifurcation diagram reveals which parameters lead to the emergence of novel solutions. (Or disappear). Plot the parameter value σ against all equilibrium values N^* to create a bifurcation diagram.

Vertical σ and horizontal N^* represent critical points in a diagram. Sink "curves" are depicted by solid lines, while source "curves" are shown by dashed lines. The purpose of the following examples is to illustrate how to draw a bifurcation diagram. Bifurcation at the coordinates (σ_0, N) only takes place under certain conditions.

$$f(\sigma_0, N^*)=0, \text{and } \left.\frac{\partial f(N,\sigma)}{\partial N}\right|_{N=N^*}=0$$

There seems to be no bifurcation:

(i) N^* will remain a sink for all σ sufficiently close to σ_0, the critical point, because the inequality $f(N^*)<0$ holds locally near the equilibrium point.

N^* will remain a source for all σ sufficiently close to σ_0, the critical point, because the inequality $f(N^*)>0$ holds locally near the equilibrium point.

8. Governing form of Trout pond's Logistic Model

The formulation of a trout pond's logistic model involves understanding fish growth dynamics, including intrinsic growth rate and carrying capacity. It predicts how fish grow rapidly initially and stabilizes as resources become limited, aiding in effective management decisions for sustainable stocking and harvesting practices to maintain ecological harmony. It is depicted a trout pond's logistic equation of the form

$$\frac{dN}{dt}=0.01N\left(1-\frac{N}{400}\right) \tag{7}$$

9. Bifurcation Analysis of the Equation (7)

Bifurcation analysis explores how parameter changes affect equilibrium solutions. In a trout pond model, a sink signifies a stable population where fish thrive sustainably, while a source indicates an unstable equilibrium, risking overpopulation. $N=400$ (a sink) and $N=0$ (a source) are the two possible solutions to the equilibrium problem (7).

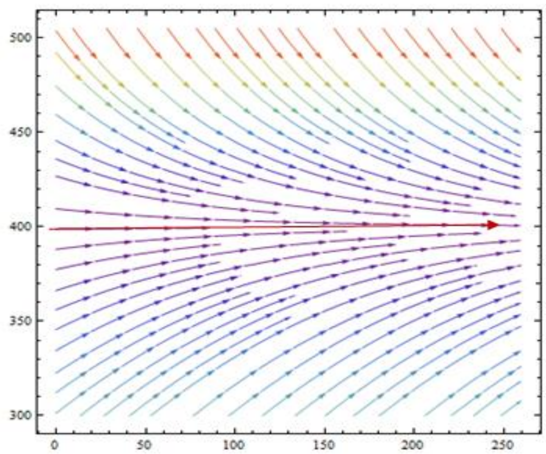

Fig. 1 Bifurcation at $N = 400$ (a sink) described by by the equation (7).

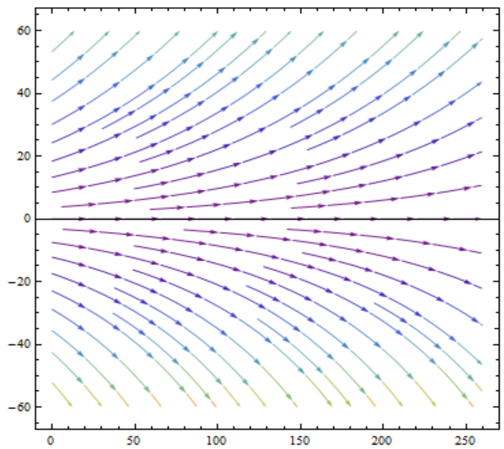

Fig. 2 Bifurcation at $N = 0$ (a source) described the equation (7).

10. Governing form of Trout pond's Logistic Model by adding the term $\frac{N}{400}$.

The equation depicts a trout pond's logistic expansion by adding the term $\frac{N}{400}$.

$$\frac{dN}{dt} = 0.01N\left(1 - \frac{N}{400}\right) + \frac{N}{400} \quad (8)$$

In equilibrium problem (8), a sink represents stability promoting population recovery and resource balance, while a source signifies instability, causing growth that diverges from equilibrium. $N = 500$ (a sink) and $N = 0$ (a source) are the two possible solutions to the equilibrium problem (8).

11. Bifurcation Analysis of the Equation (8)

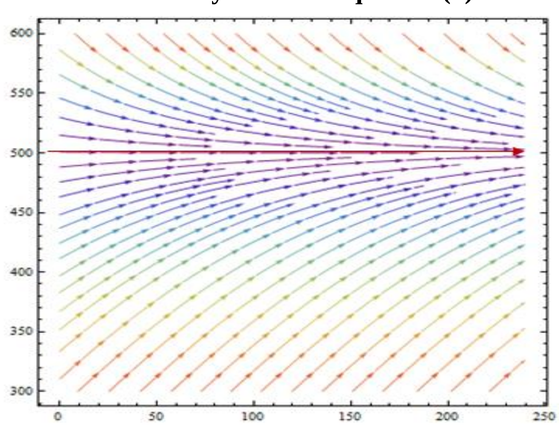

Fig. 3 Bifurcation at $N = 500$ (a sink) described by the equation (8).

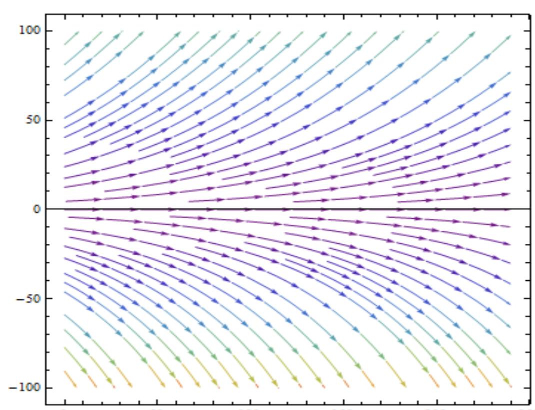

Fig. 4 Bifurcation at $N = 0$ (a source) described by the equation (8).

In the context of equilibrium problem (8), a sink represents a stable solution where the system naturally returns to equilibrium, promoting sustainability. Conversely, a source is an unstable solution that leads to divergence from equilibrium, potentially causing overpopulation or depletion of resources.

12. Governing form of Trout pond's Logistic Model with Harvesting

The trout pond's logistic model with harvesting combines population growth dynamics and sustainable fishing practices. It helps manage fish populations effectively, ensuring ecological health and economic benefits through informed stocking and harvest strategies. Let's pretend for a moment that assume $r = 0.01$ and H = 3/4 tons of fish out of pond every year. After that, the formula (1) becomes

$$\frac{dN}{dt} = 0.01N\left(1 - \frac{N}{400}\right) - H \qquad (9)$$

Where $H = \frac{3}{4}$ [Vladimir Dobrushkin, Mathematica, Part 1.2.]

Where *N* is the population size, *r* is the intrinsic growth rate, *K=400* is the carrying capacity, and *H (N)* is the harvesting function.

13. Bifurcation Analysis of the Equation (9)

$N = 300$ (a sink) and $N = 100$ (a source) are the two possible solutions to the equilibrium problem (9) from Fig. 5. When $H = \frac{7}{8}$ then $N = 270$ (a sink) and $N = 130$ (a source) are the two possible solutions to the equilibrium problem (9) from the Fig.6.

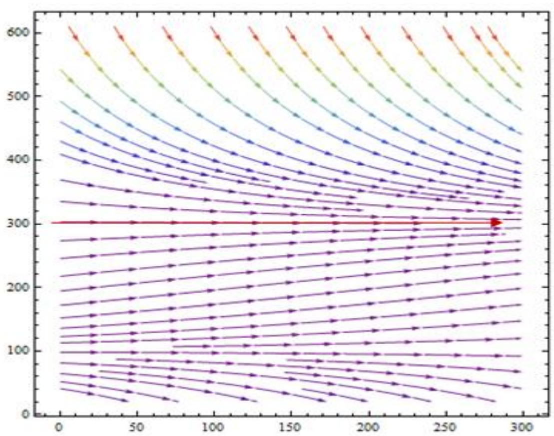

Fig. 5 Bifurcation at $N = 300$ (a sink) and $N = 130$ (a source) described by the equation (9).

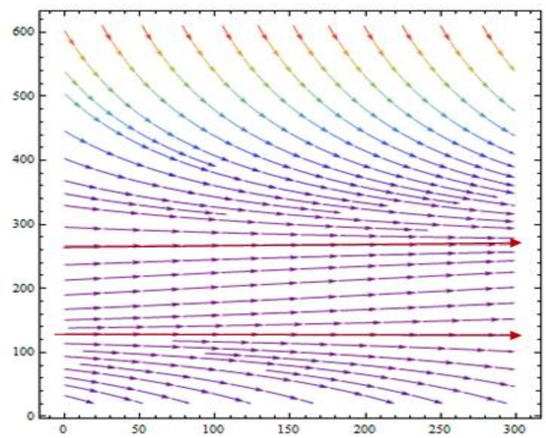

Fig. 6 Bifurcation at $N = 270$ (a sink) and $N = 100$ (a source) described by the equation (9).

When $H = \frac{3}{8}$ then $N = 358$ (a sink) and $N = 42$ (a source) are the two possible solutions to the equilibrium problem (9) from Fig. 7 and when $H = \frac{2}{3}$ then $N = 315$ (a sink) and $N = 85$ (a source) are the two possible solutions to the equilibrium problem (9) from Fig. 8.

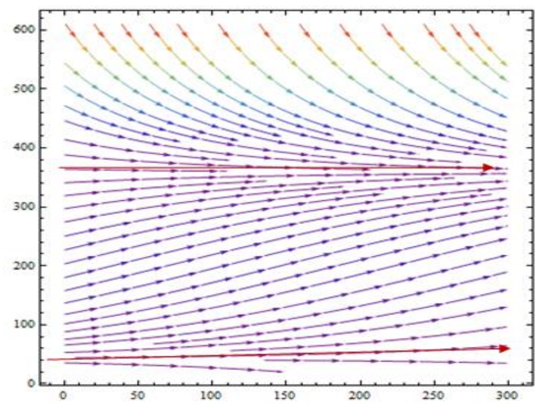

Fig. 7 Bifurcation at $N = 358$ (a sink) and (a source) described by the equation (9).

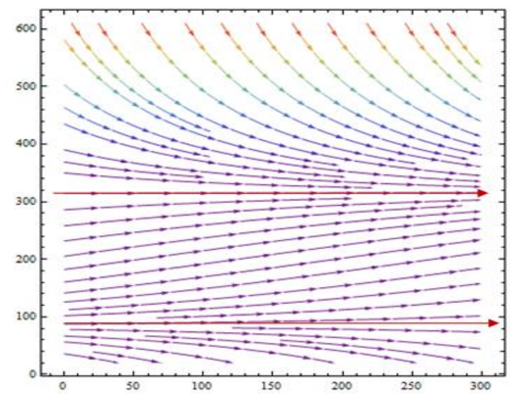

Fig. 8 Bifurcation at $N = 315$ (a sink) $N = 85$ (a source) described by the equation (9).

14. Formulation of Trout pond's Modified Logistic Model with Harvesting

Let's pretend for a moment that let H = 3/4 tons of fish out of pond every year.After that, the formula (8) becomes

$$\frac{dN}{dt} = 0.01N\left(1 - \frac{N}{400}\right) + \frac{N}{400} - H \quad (10)$$

Where H = 3/4 [Vladimir Dobrushkin, Mathematica, Part 1.2.]

15. Bifurcation Analysis of the Equation (10)

$N = 430$ (a sink) and $N = 70$ (a source) are the two possible solutions to the equilibrium problem (10) from the Fig. 9 and When $H = \frac{7}{8}$ then $N = 415$ (a sink) and $N = 85$ (a source) are the two possible solutions to the equilibrium problem (10) from the Fig. 10

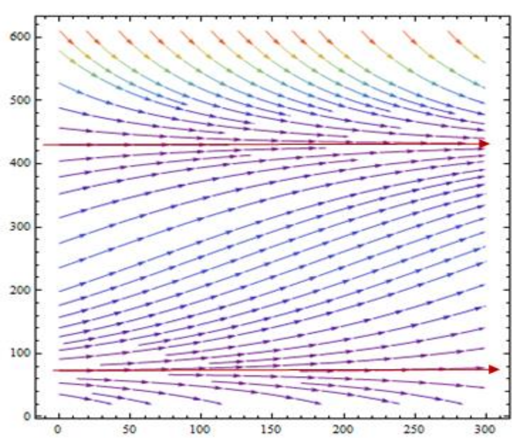

Fig. 9 Bifurcation at $N = 430$ (a sink) and $N = 85$ (a source) described by the equation (10).

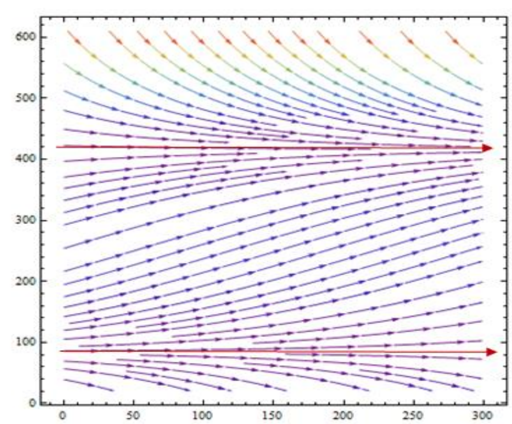

Fig. 10 Bifurcation at $N = 415$ (a sink) and $N = 70$ (a source) described by the equation (10).

When $H = \frac{3}{4}$, then $N = 468$ (a sink) and $N = 32$ (a source) are the two possible solutions to the equilibrium problem (10) from the Fig. 11. When $H = \frac{2}{3}$ then $N = 440$ (a sink) and $N = 60$ (a source) are the two possible solutions to the equilibrium problem (10) from the Fig.12.

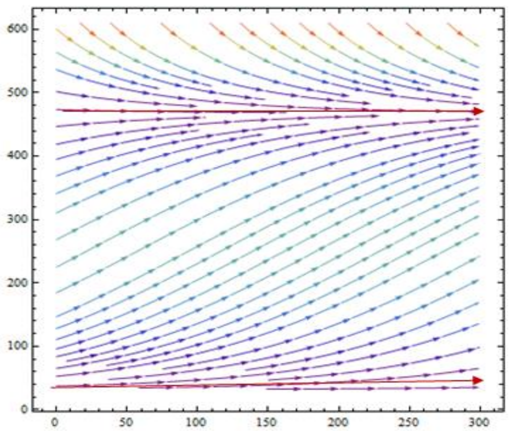

Fig. 11 Bifurcation at $N = 468$ (a sink) and $N = 32$ (a source) described by the equation (10).

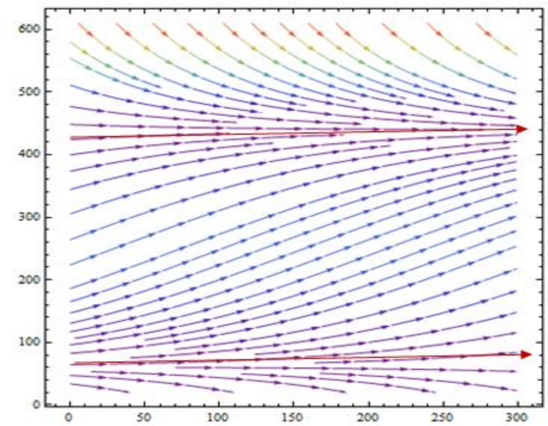

Fig. 12 Bifurcation at $N = 440$ (a sink) and $N = 60$ (a source) described by the equation (10).

It is experimented with allowing H = 20 fishermen to use the pond. The harvesting model can be written as from the equation (11)

$$\frac{dN}{dt} = 0.01N\left(1 - \frac{N}{400}\right) - 20 \qquad (11)$$

The points of equilibrium can be calculated by solving the corresponding quadratic equation.

$$0.01N\left(1 - \frac{N}{400}\right) - 20 = 0$$

$$N = 200 \pm 871.78i$$

Consequently, the quadratic equation does not have any true solutions, and there are no solutions that preserve equilibrium. Additionally, for any value of N, $\frac{dN}{dt} < 0$. This means that eventually, no matter how many fish were in the pond to begin with, the trout population will collapse due to overfishing. For a certain value of H, the quadratic equation has a solution.

$$0.01N\left(1 - \frac{N}{400}\right) - H = 0$$

In order to pick up on two crucial facts

$$N_{1,2} = 200\left(1 \pm \sqrt{1 - H}\right) \qquad (12)$$

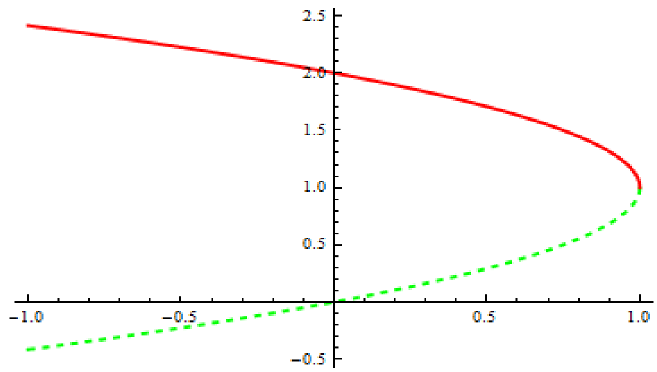

Fig. 13 The logistic equation bifurcation diagram for the constant harvesting scenario (12).

If $H < 1$ then unstable source $N_1 = 200\left(1-\sqrt{1-H}\right)$ is contrasted by the asymptotically stable sink $N_2 = 200\left(1+\sqrt{1-H}\right)$.

When H is greater than 1, the population inevitably collapses regardless of how things started.

$$\frac{dN}{dt} = 0.01N\left(1-\frac{N}{400}\right)+\frac{N}{400}-H \quad (13)$$

Where $H = 20$

$$0.01N\left(1-\frac{N}{400}\right)+\frac{N}{400}-20=0$$

The points of equilibrium can be calculated by solving the corresponding quadratic equation.

$N = 250 \pm 858.778i$

In order to pick up on two crucial facts

$$N_{1,2} = 250\left(1\pm\sqrt{1.56-H}\right) \quad (14)$$

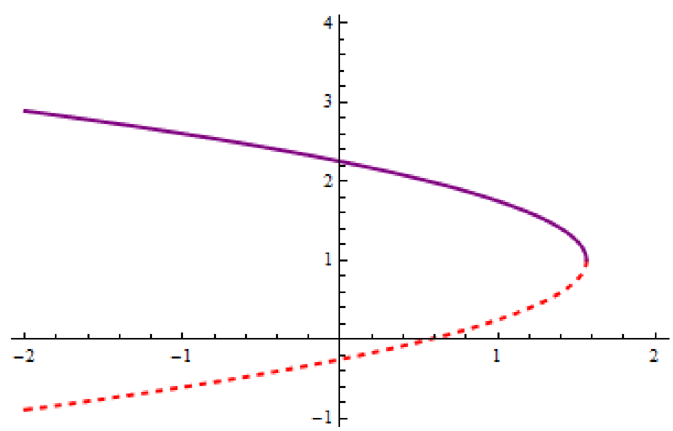

Fig. 14 The logistic equation bifurcation diagram for the constant harvesting scenario (14).

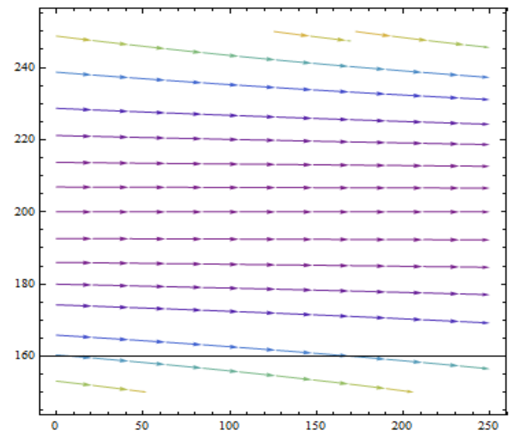

Fig. 15 Trajectories as $H = 1$ described by the equation (14).

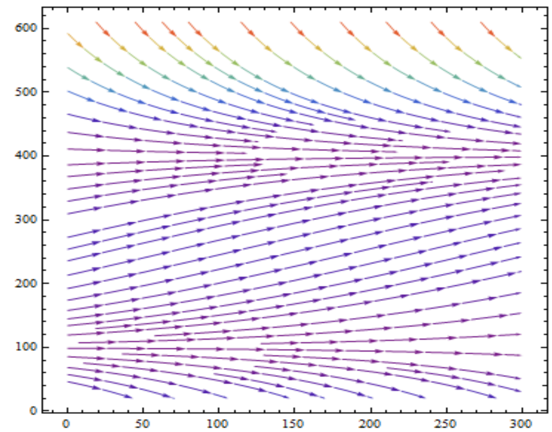

Fig. 16 Trajectories as $H = 1$ described by the equation (14).

16. Results and Discussions of Logistic Harvesting

If the population falls below 1, extinction becomes inevitable unless intervention occurs. This could correspond to a critical population threshold in the differential equation modeling the fish population, below which the growth rate cannot compensate for losses due to natural causes, predation, or fishing. If $N(t) < N_{crit}$ that is 1 fish, the population's growth becomes insufficient to compensate for losses (due to fishing or natural mortality), leading to extinction. Restocking or halting fishing ensures N (t) stays above this threshold. A slight increase in H could shift the equilibrium population below Ncrit, causing the collapse. Conversely, reducing H may stabilize the population. No harvesting if $H_0 = 0$**,** then the population stabilizes at the carrying capacity; K. Moderate harvesting $H < \frac{rK}{4}$ then the system reaches a sustainable equilibrium population aboveN_{crit}. Overharvesting if $H < \frac{rK}{4}$ then the extinction becomes inevitable as the population cannot replenish. If there are less than one hundred fish in the pond, all of them will die unless fishing is banned or the pond is restocked. The behavior of the solutions to the differential equation can be drastically changed by making a small adjustment to H. The fish population will collapse by an order of magnitude if H is increased by 1.Two equilibrium solutions become one, and eventually none, as H is increased. H = 1 is the precise moment of this shift. For the specified logistic equation, we observe a bifurcation at H = 1.

17. Conclusion

The significance of bifurcation analysis in comprehending pond trout populations in logistic harvesting models is highlighted in this paper. According to the results, alterations in harvesting rates have the potential to greatly affect population stability, thereby triggering crucial thresholds that, if not handled correctly, could cause ecological collapse. The identification of these thresholds brings attention to the fine equilibrium that is necessary for the maintenance of healthy trout populations. Policymakers and fishery managers must use these findings to promote sustainable harvesting strategies that protect trout populations over the long term without compromising ecological integrity or meeting economic demands. Research shows that pond fish populations are most at risk when their numbers fall below 100. The population is on the brink of extinction unless something is done, such a fishing restriction or restocking initiatives. Importantly, population dynamics can undergo dramatic shifts with even a little rise in the harvesting rate H, eventually leading to a bifurcation at H=1. The convergence of equilibrium solutions causes the extinction of the population at this crucial point. In order to keep fish populations sustainable and stop ecological degradation in aquatic systems, proactive management measures are needed. To maintain ecological balance in the trout pond, it is helpful to understand the logistic model in order to make educated management decisions on stocking rates and harvesting tactics. Insightful management approaches are crucial for the future of trout in pond habitats, as these findings add to the growing body of literature on population ecology and the sustainability of natural resources.

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