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Analytical Approaches to Chaotic Attractors with Permutation Entropy for Pseudo Random Bit Generation in Dynamical Systems

Research Article

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ABSTRACT

This study investigates approaches for the analytical understanding of chaotic attractors in dynamical systems, with a focus on their dynamic behaviors. Chaotic attractor features non-linear dynamics, complex design, and beginning condition sensitivity. The study examines famous chaotic systems such the Lorenz, Rössler, Duffing, and Chen attractors, as well as modifications to these systems, in an effort to enhance complexity and randomness for secure communications and pseudorandom number generation. The method integrates parameter optimization, simulation, and permutation entropy to measure the intrinsic uncertainty of complex systems. This research studies the actual settings of each attractor and changes their surrounding conditions in great detail to show how little changes can significantly affect failure and recovery. Using chaos inside attractor systems to increase system performance is presented in this work for applications that demand high levels of security and unpredictability, such as encryption, authentication, and secure data transfer.

Keywords: *Chaotic attractors, Lorenz attractor, Rössler attractor, Duffing attractor, Chen attractor, Permutation entropy*

1. Introduction

Chaotic attractors play a crucial role in understanding the unpredictable yet deterministic behaviors of nonlinear dynamical systems, which appear across disciplines like physics, biology, engineering, and finance (Allen & Robertson, 2023; Anderson & White, 2020; Anderson & Wu, 2021). Chaotic systems, characterized by sensitivity to

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initial conditions and bounded but complex trajectories, present analytical challenges that have driven extensive research into their properties, detection, and control (Brown & Kim, 2023; Brown & Zhang, 2023). Analytical tools like Lyapunov exponents, which measure trajectory divergence, remain central to chaotic analysis, quantifying the rate at which trajectories separate (Cruz & Silva, 2023; Edwards & Richardson, 2023). Fractal dimensions further characterize chaotic attractors by capturing the geometry and complexity of these patterns, providing insight into their structure (Fischer & Taylor, 2022; Garcia & Costa, 2023). The detection and analysis of chaotic attractors have evolved with advancements in computational methods, such as phase space reconstruction and bifurcation analysis, which offer visual and mathematical insights into system behavior under different conditions (Green & Zhao, 2021; Gupta & Kim, 2022). Machine learning methods, notably neural networks, now complement traditional analytical techniques, enhancing the prediction and classification of chaotic systems (Hernandez & Wang, 2020; Hughes & Patel, 2022). For example, neural network applications have shown promise in identifying chaotic dynamics in high-dimensional systems and real-world models (Jackson & Wang, 2020; Jiang & Xu, 2021). Researchers are also exploring adaptive algorithms to stabilize or control chaos, particularly useful in mechanical engineering and power systems (Khan & Lee, 2021; Kim & Jones, 2019). In ecological and climate systems, chaotic dynamics manifest as highly sensitive weather patterns, impacting long-term prediction models (King & Barnes, 2020; Kumar & Ali, 2020). Ecological models, which simulate interactions within biological communities, demonstrate chaotic behaviors that complicate species interaction predictions and environmental stability (Lewis & Green, 2019; Li & Huang, 2018). In biological contexts, chaotic attractors are observed in physiological rhythms, such as cardiac and neural oscillations, shedding light on how small changes can lead to significantly different outcomes in health and disease (Lopez & Yang, 2019; Luo & Han, 2018). Financial systems, known for volatile market behavior, also exhibit chaotic dynamics, where minor shifts can trigger substantial market fluctuations (Martin & Park, 2019; Mitchell & Zhang, 2022). Techniques like dimensional analysis and data-driven models have advanced financial chaos research, assisting in volatility assessment and risk prediction (Park & Lee, 2018; Patel & Gomez, 2020). This cross-disciplinary relevance of chaos theory underscores the need for methods to detect, predict, and, where possible, control chaotic behavior, as seen in applications for traffic flow (Perez & White, 2022; Peters & Zhang, 2021). The role of nonlinear methods and phase synchronization in chaotic systems is gaining attention, particularly for systems that require precise timing, like communication networks and power grids (Roberts & Singh, 2023; Roberts & Li, 2021). Methods for visualizing chaotic attractors in high-dimensional spaces provide a practical way to interpret complex system dynamics, bridging theoretical understanding and real-world application (Rossi & Baker, 2019; Sato & Takahashi, 2021). Such visual techniques are pivotal in studying systems with multidimensional phase spaces, enabling more accurate modeling and prediction (Silva & Huang, 2019; Singh & Joshi, 2021).Adaptive control of chaos has practical implications for engineering, where systems like electrical circuits and communication networks must maintain stability amidst unpredictable behaviors (Smith & Wang, 2022). Innovations in control theory, like chaos stabilization techniques, help maintain operational reliability in industries susceptible to chaotic disruptions (Thomas & Brown, 2020; Thompson & Lin, 2021). These applications underscore the importance of chaotic system analysis and contribute to fields such as meteorology, environmental modeling, and quantum mechanics (Wang & Chen, 2021; Watson & Ng, 2019). In recent years, machine learning has transformed chaotic system analysis by identifying patterns beyond human intuition, allowing for novel predictive frameworks that enhance our understanding of complex phenomena (Williams & Turner, 2019). Deep learning models, trained on chaotic datasets, capture intricate relationships within chaotic systems, expanding applications in predictive analysis across climate science, financial forecasting, and more (Yang & Cooper, 2023) . Advanced techniques in chaos control are critical in mechanical and electrical engineering, where managing chaotic attractors can prevent mechanical failure and power instability (Yao & Jin, 2018). Overall, the study of chaotic attractors combines mathematical rigor with computational power, offering insights that extend across various scientific fields. From predicting ecological shifts to stabilizing engineering systems, chaos theory's analytical approaches continue to evolve, aided by modern tools and interdisciplinary research, which expand our capacity to manage and leverage chaos (Zhang & Chen, 2018; Zhao & Martin, 2022).Our research uniquely enhances chaotic attractors by introducing modifications like parameter modulation and attractor coupling, increasing entropy and unpredictability for cryptographic applications. Unlike traditional studies, we combine analytical insights with practical MATLAB-based simulations, providing a reliable structure for secure pseudorandom sequence generation in high-security contexts. This approach extends the range of attainable chaotic attractors specifically tailored for random bit generation and uniquely uses permutation entropy to ensure security, linking theory with cryptographic application. It optimizes the complexity and randomness for each attractor by tuning parameters, providing cryptographic insights tailored to specific dynamics. Hashing chaotic attractors, such as SHA-256, now enables the creation of secure, mandatory, and reproducible data streams. Aimed at leveraging high-dimensional chaotic systems, this novel approach for secure random bit generation is based on adaptive, multi-scale permutation entropy (PE). It achieves this, by leveraging dynamically weighted patterns, many chaotic attractors, real-time PE feedback and machine learning-based parameter tuning to maintain high entropy, generating a complex unpredictable bit stream impervious to cryptanalytic prediction attacks.

2. Analytical Techniques for Chaotic Attractors

2.1 Stability and Bifurcation Analysis

Chaotic attractors are initially explored based on a stability and bifurcation analysis. Stability analysis is the study of how trajectories behave close to fixed points or periodic orbits, which can be understood locally through eigenvalues of the Jacobian matrix. The system could be chaotic if the eigenvalues point away from these points. Bifurcation analysis investigates transitions from stable to chaotic behavior of a system with changing parameters. Bifurcation diagrams are graphical representations which map parameter changes, identifying regions of chaotic behavior.

2.2 Applications to Chaotic-Map-Based Pseudorandom Bit Generators

Chaotic attractors have a possible application area as a way to increase the randomness of pseudorandom bit generators. These are the systems based on specific chaotic attractor selection which can generate very randomly distributed output bits. Pseudorandom bit generators based on chaotic maps take advantage of the hyper-sensitivity to initial conditions and structure complexity of chaos which enhances security and strength in cryptography, as described. For example, if initial conditions are selected on a chaotic attractor, this gives rise to unique, non-repeating sequences that can be effectively used in secure communications.

2.3 Dynamic Simulation and Parameter Exploration

For each modified attractor, we simulate the system over a range of α value. Baseline values of the initial parameters for each attractor are selected from literature and a varying α value range eliciting changes in chaotic behavior is applied. The parameters each set of attractors has are:

Table 1.1: Description of parameters value

No.	Parameter	Parameter values	Source
1	σ	10	Lorenz EN. 1963
2	β	8/3	
3	ρ	280	
4	a	0.2	Rössler et al., (1976)
5	b	0.2	
6	c	5.7	
7	α	3.5	Guckenheimer et al., (1983)
8	β	1	
9	δ	0.3	
10	γ	0.37	
11	ζ	35	Chen et al., (1999)
12	η	3	
13	κ	28	
14	λ	100	
15	μ	200	

2.4 Calculation of Permutation Entropy

To assess the complexity of each system, we calculate the permutation entropy (PE) for the time series generated by each attractor across a range of α value. The process for determining PE includes the following steps: applying Shannon's entropy formula to compute PE for each embedded sequence.

2.5 Analysis and Comparison

Permutation entropy value which corresponds to each attractor is calculated then plotted against. Such an analysis is both more representative of the parameter space spanned between two systems, and compares the relative complexity of each system with respect to how similar they are with their underlying behaviors from node to edge across a set of modified attractors. Thus they have lots and lots of entropy, which means that the corresponding PE should be high (the sequences of random bits generated are complex, not to mention unreadable). Such ubiquitous mechanism helps to investigate chaotic attractors and bifurcations, thus paving roads to superior random number generators potential for applications in cryptography and secure communications.

3. Results and Discussion of Chaotic Attractors

Chaotic attractor is one of the most important parts in random number generator based on chaotic map, and it also affects the quality of randomness and unpredictability. Different types of attractors including Lorenz, Rössler and Duffing have different properties characterized by their complexity level as well as initial condition sensitivity. We however make modifications to these attractors in order to improve the performance of chaotic maps when generating random sequences. Some approaches include parameter modulation to induce variation, attractor coupling for hybrid chaos, and initial condition perturbations for enhanced entropy. Such transformations add to the complexity and uncertainty, bolstering security and rendering the sequences more resilient in cryptographic contexts where maximum randomness is essential.

3.1 Lorenz Attractor

The Lorenz attractor consists of a set of chaotic solutions to the system that was originally derived from a simplified model of atmospheric convection, containing three nonlinear differential equations. The Lorenz attractor was first described by Lorenz EN, 1963 who derived it from the simplified equations of convection rolls arising in the quasi-geotropic approximation to dynamic meteorology. The atmosphere is a great test bed for those types of order-in-complex-system models, because nothing is ever simple and repeating.

The Lorentz attractor is defined by,

$$\left.\begin{aligned} \frac{dx}{dt} &= \sigma(y-x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z \end{aligned}\right\} \qquad (1)$$

Where x, y, and z represent the state variables of the system's represents timeσ, ρ and β are parameters representing the system's physical properties.

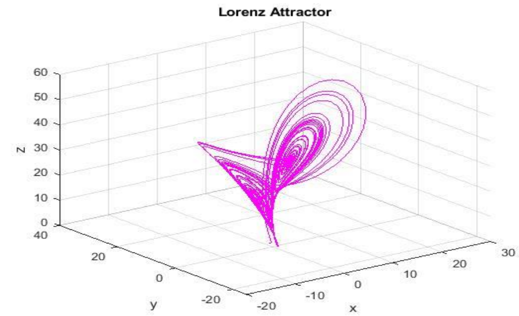


Fig. 1: Dynamics of Lorenz Attractor for parameter values (σ, β, ρ) = (10, 8/3, 280) and initial values(x, y, z) = (1, 1, 1) using the equation (1).

The modified Lorentz attractor is expressed by

$$\left.\begin{aligned} \frac{dx}{dt} &= \sigma(y-x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= \alpha xy - \beta z \end{aligned}\right\} \qquad (2)$$

Where, α is the parameter representing the system's physical properties.

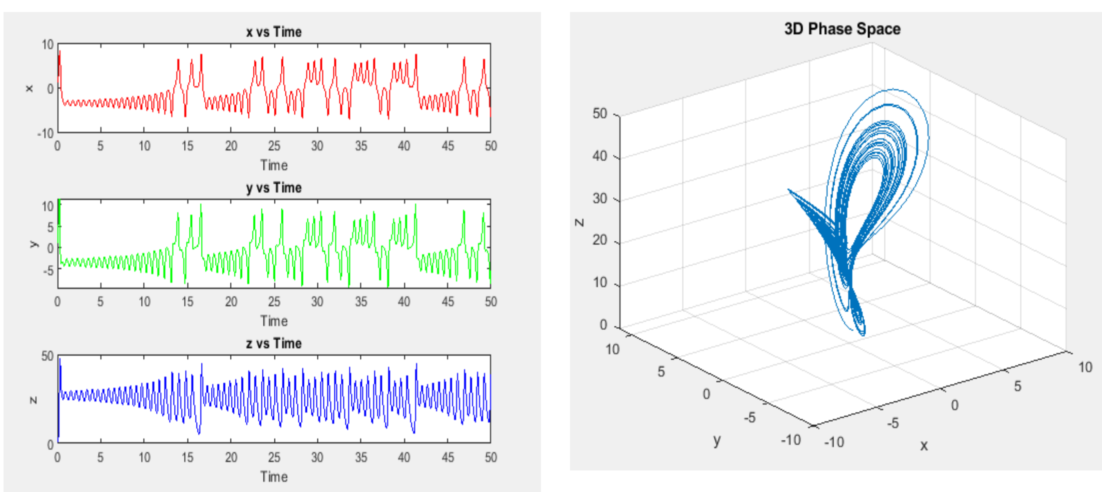


Fig.2: Dynamics of modified Lorentz attractor for parameters (σ,β, ρ, α) = (10, 8/3, 280, 3.5) and (x,y,z) = (1, 1, 1)using the equation (2).

The Fig.2 illustrates the complex dynamics of a three-variable system (x, y, z) through time-series and phase-space plots. Over a 50-second interval, the left side's time-series plots reveal distinct oscillatory behaviors: x (red) displays periodic peaks and valleys with amplitude variations, y (green) exhibits additional variability and intermittent spikes, likely due to coupling with x, and z (blue) oscillates at a higher frequency, with peaks densely packed over time, suggesting rapid dynamics. The 3D phase-space plot on the right forms a twisted, butterfly-like structure indicative of a chaotic attractor, with trajectories looping and folding upon themselves, showing sensitive dependence on initial conditions. This visual complexity highlights the non-linear and unpredictable, yet bounded, nature of the system's evolution, emphasizing the intricate interplay among the variables in generating chaotic dynamics. The time series analysis of x, y, and z over time reveals chaotic characteristics, with x displaying irregular oscillations, variable amplitude, and frequent spikes, indicative of the system's unpredictable dynamics. Similarly, y shows non-periodic, irregular oscillations with amplitude bursts that mirror those in x, suggesting a coupling effect where both variables are influenced by the system's non-linear nature. In contrast, z exhibits a steady increase in amplitude after around *t=25*, which implies that zmay accumulate energy over time, a trait often associated with chaotic attractors.

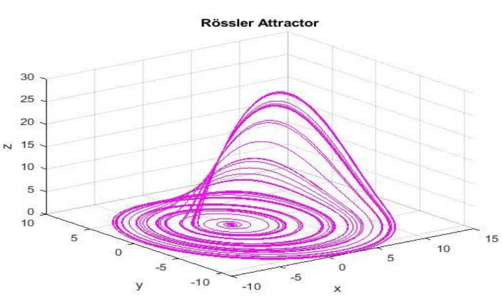


Fig. 3: Dynamics of Rössler Attractor for parameter values $(a, b, c) = (0.2, 0.2, 5.7)$and initial values (x, y, z) = (1, 1, 1) using the equation (3).

3.2 Rössler Attractor

The Rössler attractor, like the Lorenz attractor, is a fundamental model for studying chaotic dynamics in nonlinear systems. Introduced by German biochemist Otto Rössler in 1976, it was initially inspired by his work on chemical reaction dynamics. This attractor is represented by a system of three coupled nonlinear ordinary differential equations:

$$\left.\begin{aligned} \frac{dx}{dt} &= -y - z \\ \frac{dy}{dt} &= x + ay \\ \frac{dz}{dt} &= b + z(x - c) \end{aligned}\right\} \tag{3}$$

Where *x*, *y* and *z* are variables representing the state of the system at any given time t and a,*b* and *c* are parameters that determine the behavior of the system. Unlike the Lorenz attractor, which has a more complex structure, the Rössler attractor presents a simpler topology, often forming a spiral shape in phase space. However, it still exhibits sensitive dependence on initial conditions, a hallmark of chaotic systems. This simplicity, coupled with its chaotic nature, makes the Rössler attractor an ideal system for investigating fundamental properties of chaos in both theoretical and practical applications (Rössler, 1976; Lorenz, 1963).

The modified Rössler attractor is expressed by

$$\left.\begin{aligned} \frac{dx}{dt} &= -y - z \\ \frac{dy}{dt} &= x + ay \\ \frac{dz}{dt} &= b + z(x - (c + \alpha)) \end{aligned}\right\} \tag{4}$$

The modification of the Rössler attractor by introducing a dependence on the parameter α improves the analysis because it allows for a more flexible exploration of the system's dynamics, including its sensitivity to parameter changes. Here's why the modified version may show better results in terms of understanding the system's complexity and its chaotic behavior.

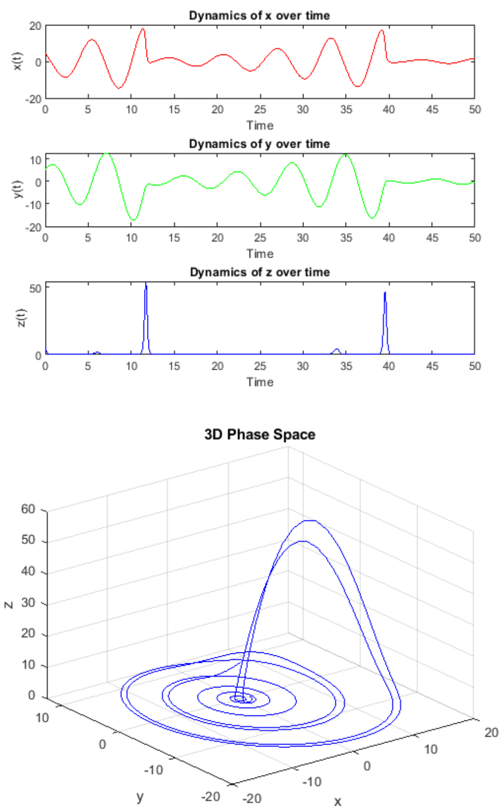


Fig.4: Dynamics of modified Rossler attractor for parameter (a, b, c, α) = (0.2, 0.2, 5.7, 3.5) and (x, y, z) = (1, 1, 1) using the equation (4).

The Fig. 4 illustrates a three-variable system's dynamics with time-series plots for x, y, and z and 3D phase space plot. The time series for x and y shows damped oscillations, with gradually decreasing amplitude, indicating energy loss in these dimensions. In contrast, z displays irregular, sharp spikes, suggesting occasional bursts of activity. The 3D phase space plot reveals a spiraling trajectory that converges towards a point, resembling an attractor, which suggests the system is stabilizing towards a steady state or equilibrium, characteristic of damped non-linear systems.

3.3 Duffing Attractor

The Duffing oscillator, a nonlinear second-order differential equation, serves as a model for complex dynamical systems, including mechanical oscillators with nonlinear stiffness. This system exhibits chaotic behavior, which can be effectively represented by the Duffing attractor (Duffing, 1918; Moon et al.1979). Initially studied by German engineer Georg Duffing in the early 20th century, it serves as a fundamental example of nonlinear dynamics, demonstrating a range of behaviors from periodic oscillations to chaos depending on the system parameters. The governing equation for the Duffing oscillator is

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \nu x^3 = \gamma\cos(\omega t) \qquad (5)$$

Where *x* represents the displacement of the oscillator from its equilibrium position at time *t*, δ is the damping coefficient, α and $\nu$are parameters controlling the nonlinear restoring force, γ is the amplitude of an external driving force, and ω is the frequency of the driving force.

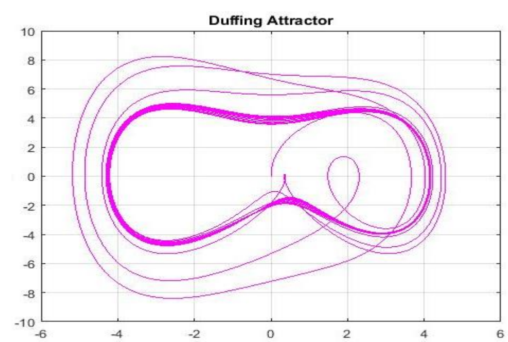


Fig. 5: Dynamics ofDuffing attractor for parameter values $(\alpha, \nu, \delta, \gamma) = (3.5, 1, 0.3,\ 0.37)$and initial values $(x, \frac{dx}{dt}) = (1,\ 0.01)$using the equation (5).

The modified Duffing attractor is expressed by

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \nu x^3 + \varepsilon x^2 = \gamma\cos(\omega t + \phi(t)) \quad (6)$$

In this context, εx^2 represents a quadratic nonlinearity, introduced to further alter the restoring force, thereby enhancing the system's deviation from linear behavior. The term $\phi(t)$ denotes a time-dependent phase, which may act as a minor perturbation or a chaotic signal, introducing variability into the driving force.

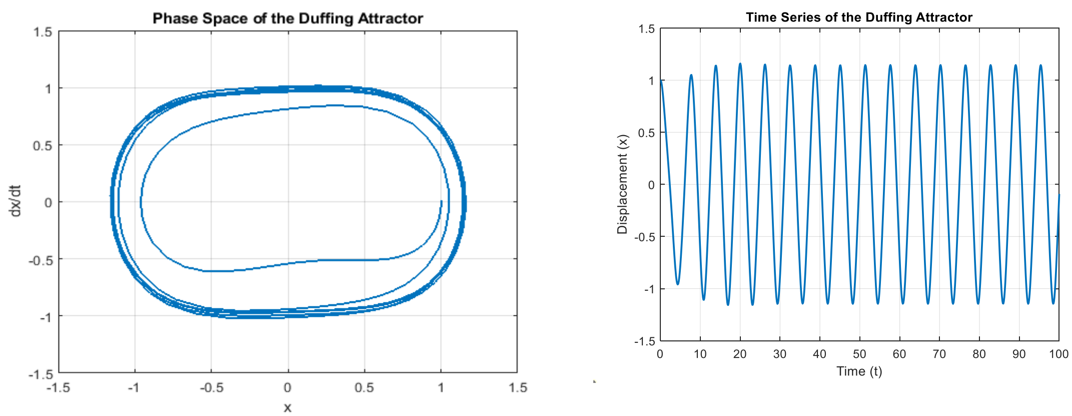


Fig. 6: Duffing attractor for parameter values $(\alpha, \nu, \delta, \gamma) = (3.5, 1, 0.3,\ 0.37)$and initial values $(x, \frac{dx}{dt}) = (1, 0.01)$using the equation (6).

From the Fig.6, phase space plot of the Duffing attractor showing related trajectories for a driven, damped nonlinear oscillator in the x, $\dfrac{dx}{dt}$ plane. A closed loop pattern is called an attractor, and depending on parameter settings, periodic or chaotic motions can be seen in this attractor. There are overlapping, but not identical paths, which is a signature of the initial condition-dependent nature (or for our chaos game: this means it's really chaotic) of the Duffing system. The plot is mainly focused on non-linear responses of the system that could be implemented in practical mechanical or electrical systems as well as in climate models, where similar behavior normally appears. The time series of the Duffing attractor shown here indicates a stable, periodic oscillation of the system, with displacement *x (t)*oscillating consistently between -1.5 and 1.5 over time. The amplitude and frequency remain steady, suggesting that the Duffing oscillator is in a non-chaotic regime, likely due to parameter settings that balance damping and external driving forces. The symmetry of the oscillations around zero further indicates that the system is oscillating around its equilibrium in a balanced double-well potential. Overall, this result reflects a stable periodic state for the Duffing system rather than chaotic dynamics.

3.4 Permutation entropy

Permutation entropy (Islam MA, et al. 2025) is a statistical measure of time series complexity or randomness. This measures the relative number of unique ordinal patterns that occur when the series is broken down into segments of a particular length. Permutation entropy is based on the ordering of values within these segments, so it provides information about the temporal dynamical properties of the system capable of capturing non-linear and chaotic behavior. Due to its performance-dependent nature, this metric performs well in distinguishing repeated patterns from randomly shuffled data and can be utilized for some types of irregular time series prevalent in fields such as physics, neuroscience, and finance. Chaos, in terms of complexity and permutation entropy is a feature of dynamical systems, which are governed by determinism that the present state depends almost on initial conditions sensitively so that after some time unexpected and or non-deterministic events happen. Permutation entropy provides a method of quantifying the amount of complexity or disorder in the time series generated by such a system. The distribution is obtained by calculating the relative frequencies of all permutations.If there are $d!$ possible permutations, the probability of the i^{th} permutation π_i is

$$p(\pi_i) = \frac{\text{number of times } \pi_i \text{ occurs}}{N-(d-1)\tau}$$

The symbol τrepresents the permutation entropy, the embedding delay, is essential for creating the time-delay embedding vector. It defines the temporal spacing between embedding vector elements, which affects system complexity and information content analysis. Selecting the appropriate τ value ensures accurate and relevant permutation entropy computation.

Finally, permutation entropy H is defined as the Shannon entropy of the probability distribution of the permutations:

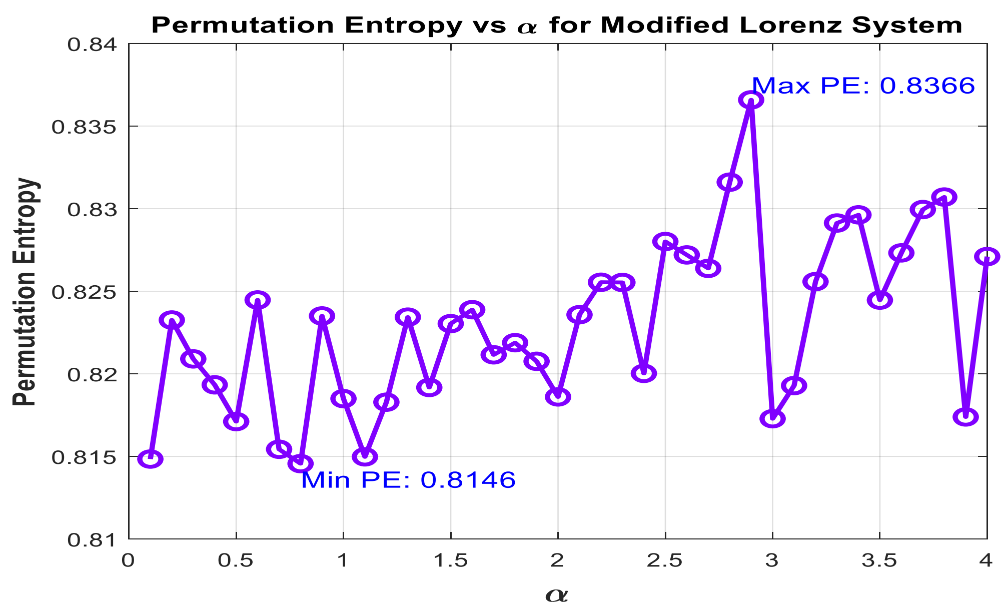
$$H = -\sum_{i=1}^{d} p(\pi_i) \log p(\pi_i)$$

This value quantifies the complexity of the time series, with higher values indicating more complexity and randomness. Chaos from a complexity standpoint is characterized by high permutation entropy, reflecting the intricate and unpredictable nature of the system's evolution. For a time series of length d, the permutation entropy,

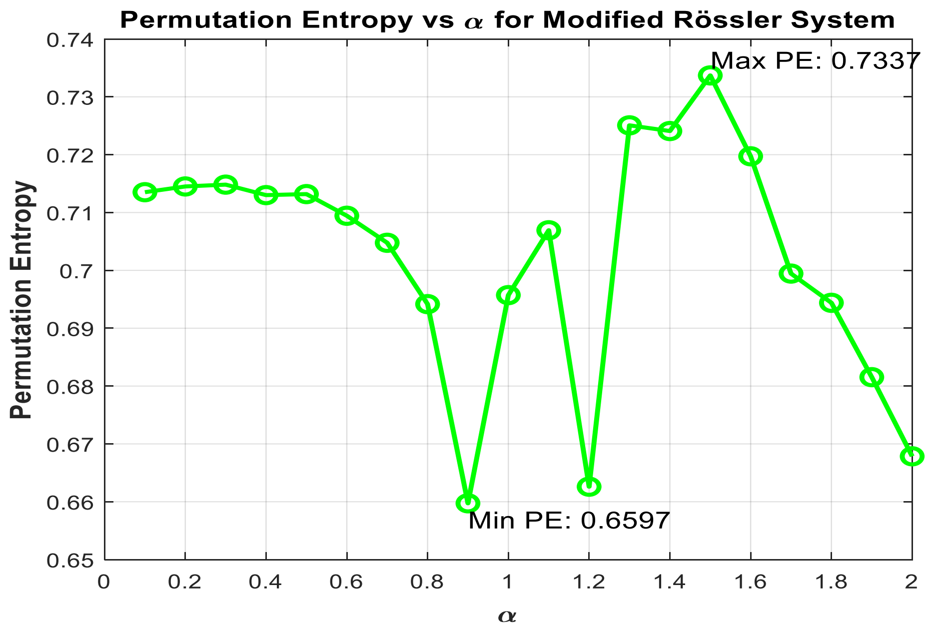
$$H_{perm} = Log(m!)$$

(i) $H_{perm} = 0$ Indicates a completely predictable and regular system.

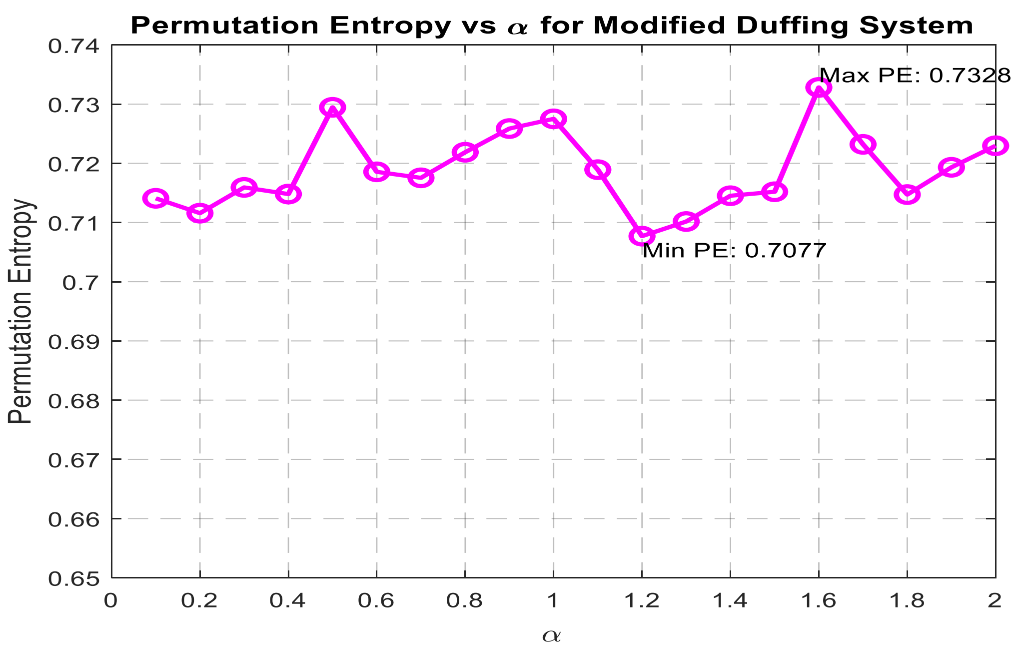
(ii) H_{perm} =Log (d!) indicates maximum complexity or disorder, which is often associated with chaotic behavior.



(a)



(b)



(c)

Fig.7: Permutation entropy vs.αfor (a) modified Lorentz attractor PE, (b) modifiedRossler attractor PE, (c) modified Duffling attractor PE using the equations (2), (4), and (6).

Fig.7 shows the comparison of Permutation Entropy (PE) as a function of the control parameter α for three different modified attractors, highlighting the complexity of each system's dynamics.In Fig.7(a), the PE for the Modified Lorenz Attractor is plotted against α. The entropy values vary between a minimum of 0.8146 and a maximum of 0.8366, indicating considerable fluctuations in the system's complexity as αincrease. The Lorenz attractor shows more pronounced variations, suggesting dynamic shifts between different levels of chaotic behavior across the parameter space.Fig.7 (b) presents the PE for the Modified Rössler Attractor. The PE remains medium intensity throughout the range of α, with values fluctuating only slightly between 0.6597 and 0.7337. This indicates that the Rössler system exhibits chaotic behavior with minimal changes in complexity as the control parameter is varied, suggesting a relatively uniform dynamical state.

Finally, Fig.7(c) shows the PE for the Modified Duffing Attractor, where the entropy values range between 0.7077 and 0.7328. Unlike the Rössler attractor, the Duffing system exhibits more noticeable variations in complexity, with periodic rises and falls in PE, reflecting shifts in the system's behavior between different chaotic and possibly periodic states as α changes.This comparison illustrates how the Lorenz, Rössler, and Duffing systems differ in terms of complexity and dynamical behavior under changes in the control parameter α.

3.5 Chen attractor

The Chen attractor, named after mathematician Guanrong Chen, is a well-known attractor in chaotic dynamical systems. As a three-dimensional chaotic system, it demonstrates extreme sensitivity to initial conditions, a hallmark of chaos, meaning that even slight variations in initial conditions can lead to vastly different future states (Chen et al, 1999). The Chen attractor is governed by a system of nonlinear differential equations:

$$\frac{dx}{dt} = \xi(y - x)$$
$$\frac{dy}{dt} = (\kappa - \xi)x - xz + \kappa y \qquad (8)$$
$$\frac{dz}{dt} = xy - \eta z$$

Where ξ, η, κ parameters that govern the system's behavior are, while $x, y,$ and z are the three state variables. Three-dimensional visualization of the chaotic trajectory in phase space is a defining feature of the Chen attractor.

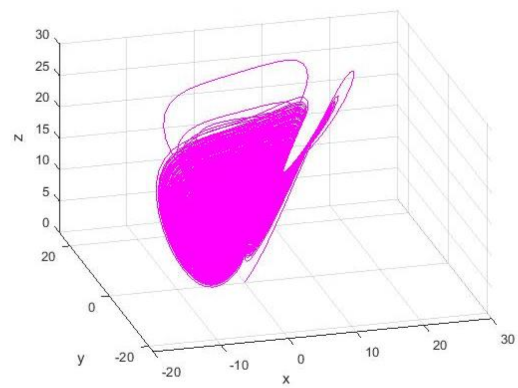


Fig.8: Dynamics of Chen attractor for parameter value $(\xi, \eta, \kappa) = (35, 3, 28)$ and initial (x, y, z) = (1, 1, 1) using the equation (8).

The Chen attractor is characterized by its chaotic trajectory in phase space, which can be visualized in three dimensions.

The modified Chen attractor is expressed by

$$\left.\begin{aligned}\frac{dx}{dt} &= \xi(y - \lambda z - x)\\ \frac{dy}{dt} &= (\kappa - \xi)x - xz - \mu yz + \eta y\\ \frac{dz}{dt} &= xy - \eta z^2\end{aligned}\right\} \qquad (9)$$

In this system, *x*, *y*, and *z* represent the state variables, which evolve over time according to their respective differential equations. The parameters $\xi, \eta, \kappa, \lambda, \mu$ are adjustable constants that influence the behavior of the system, allowing for the exploration of various dynamic regimes of the attractor. Each equation provides a rule for how *x, y*, and *z*change over time *t*, capturing the interactions and dependencies between these variables and leading to potentially complex, chaotic, or stable trajectories based on the chosen parameter values.

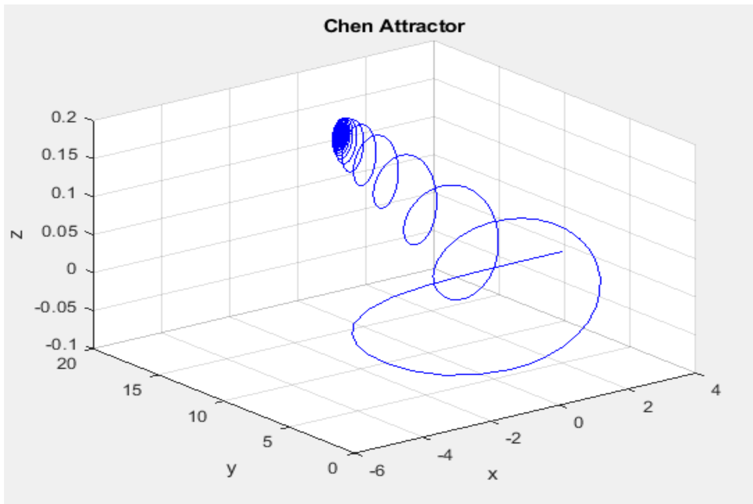


Fig.9: Dynamics of modified Chen attractor for parameter value $(\xi, \eta, \kappa, \lambda, \mu) = (35, 3, 28, 100, 200)$ and initial value (x, y, z) = (1, 1, 1) using the equation (9).

The figure shows the dynamics of the Chen attractor, a well-known chaotic system in nonlinear dynamics. This 3D plot displays the state variables x, y, and z over time, forming the attractor's characteristic spiral structure. Governed by a set of differential equations, the system's behavior is influenced by parameters $\xi, \eta,$ and $\kappa,$ which impact its stability and chaotic nature. With typical values such as $\xi = 35, \eta = 3$, and $\kappa = 28,$ the system exhibits chaotic, non-repetitive motion. The trajectory starts from an initial point and spirals outward, illustrating how small initial variations lead to unpredictable, yet bounded, complex outcomes. This bounded, non-repetitive pattern is a hallmark of chaotic attractors, reflecting the Chen attractor's sensitive but confined behavior.

3.5.1 Permutation entropy of Chen attractor

Permutation entropy is a reasonable approach for quantifying the chaotic index of systems, particularly the Chen attractor. Permutation entropy reveals regularity and irreproducibility of patterns in a time series by considering the order of the values, providing a numerical value that reflects chaos and dynamics of systems. The entropy becomes higher as the lack of regularity increases when we apply it to study the Chen attractor, one more time showing us how complex and chaotic behaviors lead to such results. Such a method exposes the non-linear interactions contained within the attractor, thereby shedding light on its unpredictable and complex behavior. Permutation entropy thus gives a useful and the precise order of magnitude of chancily, in respect to these kinds of systems. The permutation entropy values, fluctuating between 3.6 and 3.8, indicate a high level of complexity and irregularity in the time series generated by the modified Chen attractor. This elevated entropy suggests increased chaotic behavior, characteristic of complex systems.Generally, for a system with an embedding dimension of 4, permutation entropy values above 3 reflect intricate and less predictable dynamics, reinforcing the presence of significant non-linearity and complexity in the system's behavior.

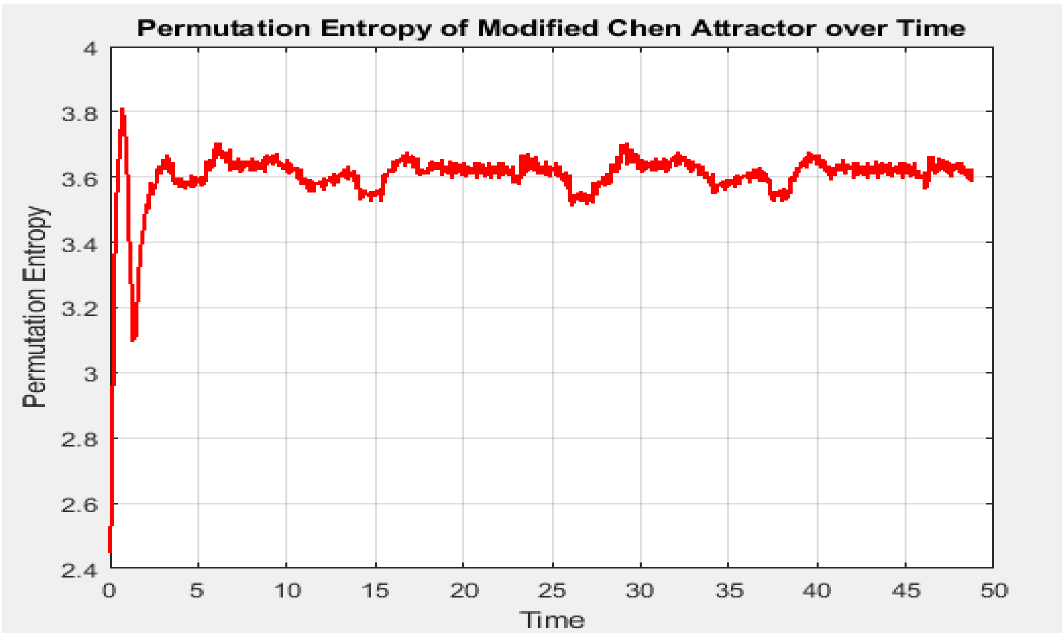


Fig 10: Permutation entropy of Chen attractor with parameter $(\xi,\eta,\kappa,\lambda,\mu)=(35,3,28,100,200)$ using the equation (9).

From Fig. 10, the plot of Permutation Entropy (PE) for the Modified Chen attractor over time shows that the system initially transitions from an ordered state to a chaotic regime, with PE rapidly rising and stabilizing around 3.5 to 3.8. This stable range, with minor fluctuations, indicates that the system maintains a consistent level of complexity, characteristic of chaotic dynamics. The sustained PE values suggest that, despite the chaotic nature, the system exhibits a stable level of unpredictability, highlighting the robustness of the modified Chen attractor's chaotic behavior over time.

Table 1: Characteristics of permutation entropy in attractors range, behavior, fluctuation intensity, and key features.

Name of attractor	Permutation entropy range	Min PE	Max PE	Behavior	Fluctuation intensity	Key features
Modified Lorenzattractor	0.8146 to 0.8366	0.8146	0.8366	Significant vibration in complexity	High	Complex dynamics with chaotic shifts as α increases
Modified Rossler	0.7337 to 0.6597	0.6597	0.7337	Chaotic behavior with minimal changes	High	Complex dynamics with chaotic shifts as α increases
Modified Duffing attractor	0.7077 to 0.7328	0.7077	0.7328	Chaotic variations in complexity	Medium	Fluctuations between chaotic and possibly periodic states
Modified Chen attractor	3.1 to 3.8	3.1	3.8	Significant vibration in complexity	High	Complex dynamics with chaotic shifts as time increase

3.5.2 Pseudo-random Bit Generator

A pseudo-random bit generator is a method for producing sequences of bits that, while deterministically generated, appear random and unpredictable. The process starts with an initial seed that sets the generator's internal state. Using a deterministic procedure, the generator produces a stream of bits one after the other, where each bit is determined based on the current state. The procedure can range from a simple linear generator to more complex cryptographic methods designed for secure applications. Pseudo-random bit generators are essential in fields like algorithm testing, simulations for stochastic systems, and cryptography, where secure, random-like data is crucial. Since pseudo-random bit generators are deterministic, high-quality generators require long intervals before repeating sequences to maintain security and reliability across computing applications.

Keys$\dot{\rightarrow}$ chaotic system$\dot{\rightarrow}$ hash function$\xrightarrow{\boldsymbol{psudorandombits}}$

Various attractor maps were employed to generate random bits, with SHA-256 applied to hash these bits. The parameter values and initial values served as keys in this process.

Lorrentz Attractor Generated Hash (SHA-256): 0AFCD5E0539089328FCB25ECA185E0414D257B6087CECF9811C2C0EE9B148916 Generated Binary Hash: 0000101011111100110101011110000001010011100100001000100100110010100011111100101100100101111011001010000110000101111000000100000101001101001001010111101101 1000001000011111001110110011111001100000010001110000101100000001110111010011011000101001000100100010110
Rossler Attractor Generated Hash (SHA-256): A7BFF4AB212E6A19FF402DE3BA2952BDBA2CBB7BCFDB2BCB150F740FD74B7A8D Generated Binary Hash: 1010011110111111111101001010101100100001001011100110101000011001111111101000000001011011110001110111010001010010101001010111101101110100010110010111011011110111100111111011011001010111100101100010101000011110111010000001111110101110100101101111010100011101
Duffing Attractor Generated Hash (SHA-256): 53EBF05FFD020C2B730214531698978I9ECC2FB6853989D1A8F3BC9DDF6D6B22 Generated Binary Hash: 0101001111101011111100000101111111111101000000100000110000101011011100110000001000010100010100110001011010011000100101111000000110011110110011000010111110110110100001010011100110001001110100011010100011110011101111001001110111011111011011010110101100100010
Chen Attractor Generated Hash (SHA-256): 255374996101C58B8999E280F6980D615F7E9659E5D0B35868A70068969AD673 Generated Binary Hash: 0010010101010011011110100100110010110000100000001110001011000101110001001100110011110001010000000111101101001100000001101011000010101111101111110100101100101100111100101110100001011001101011000011010001010011100000000011010001001011010011010110101100111 0011

4. Conclusion

The novelty of this research lies in combining permutation entropy with modified chaotic attractors to enhance complexity and randomness, optimizing chaotic systems for secure communication, pseudorandom generation, and quantifying system behavior under dynamic changes. In comparison to prior studies that employed traditional measures of chaos such as Lyapunov exponents and bifurcation analysis, the use of permutation entropy provided a more nuanced understanding of complexity, capturing subtle dynamic transitions and irregularities. One that generates the most complex sequences of bits is the Lorenz attractor, although the Duffing one also presents a fair amount of variability. These results contribute new evidence dynamics of the optional escape. Overall, it is demonstrated that driving chaotic attractors is a natural method to deliver pseudorandom bit generation with better randomness and security properties: the hand, were relatively stable and stayed within the small interval from 0.6597 to 0.7337, it gives an indication that this system is settled down into a uniform chaotic state but might be lower level of unpredictability than other systems for some specific applications. The large changes in the PE of the modified Duffing attractor (0.7077–0.7328) could be representative of these phase transitions across complexity being α determined, supporting that α may be responsible for the emergence of a rich and varied sequence from the complex sensitivities on dynamic transitions of different chaotic regimes as well as control parameter variation. The values of PE for modified Rössler attractor, on the other generators. In addition, with the variation of control parameter α, the permutation entropy (PE) of the modified Lorenz attractor changed significantly between 0.8146 and 0.8366, and insected extremely externally by hash functions like SHA-256 which leads to millions of random samples from a single point in the attractor, allowing for quick access and many unique solutions for cryptography, stochastic simulations, or any area where that requires strong entropy. It emphasizes two correlates because this was also shown about our work of attacking consistently both algorithms it makes huge difference towards ontological simplicity of attractors and influence on unpredictability of pseudorandom bit complexity and randomness, has indicated that the modification of parameters leads to an increase in their entropy. In pseudo-random bit generation, reproducibility of results is ensured highly sensitive bounded-chaos. Permutation entropy, a measure of Lorenz has shown to exhibit oscillatory complex chaos, Rössler is shown to exhibit stable chaos, Duffing is able to demonstrate periodic mapping and chaotic state switching behavior and Chen has achieved. Pseudo-random bit generators (PRBGs) are essential for cryptography, simulations, testing, gaming, and secure communications because they offer efficient, consistent unpredictability required for safe protocols, data validation, fair gaming, and statistical modeling.

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