# Characterizations of relative n-annihilators of nearlattices

**Dr. M. S. Raihan**

**Associate Professor**

**Department of Mathematics**

**Rajshahi University**

# Characterizations of relative n-annihilators of nearlattices

**Abstract:** In this paper we have introduced the notion of relative n-annihilators around a fixed element n of a nearlattice S which is used to generalize several results on relatively nearlattices. We have also given some characterizations of distributive and modular nearlattices in terms of relative n- annihilators.

**Keywords:** Annihilator, Relative annihilator and Relative n-annihilator.

**AMS Subject classifications (2001):** 06A12, 06A99, 06B10.

**Introduction.**

Relative annihilators in lattices and semi-lattices have been studied by many authors including [1], [2], [3] and [4]. Also [5] has used the annihilators in studing relative normal lattices. In this paper, we introduce the notion of relative annihilators around a fixed element n of a nearlattice S which is used to generalize several results on relatively nearlattices.

For $a, b \in S$, $< a, b >$ denotes the relative annihilator, that is

$< a, b > = \{x \in S : x \wedge a \leq b\}$. In presence of distributivity, it is easy to show that each relative annihilator is an ideal. Also note that $< a, b > = < a, a \wedge b >$. For detailed literature on this see [1] and [4]. Again for $a, b \in L$, where L is a lattice, recall that $< a, b >_d = \{x \in L : x \vee a \geq b\}$ is a relative dual annihilator. In presence of distributivity of L, $< a, b >_d$ is a dual ideal (filter).

In case of a nearlattice it is not possible to define a dual relative

annihilator ideal for any a and b. But if n is an upper element of S, then $x \vee n$ exists for all $x \in S$ by the upper bound property of S. Then for any $a \in (n]$, we can talk about dual relative annihilator ideal of the form $< a, b >_d$ for any $b \in S$. That is, for any $a \leq n$ in S, $< a, b >_d = \{x \in S : x \vee a \geq b\}$.

For $a, b \in S$ and an upper element $n \in S$,

we define, $< a, b >^n = \{x \in S : m(a, n, x) \in < b >_n\}$

$$= \{x \in S : b \wedge n \leq m(a, n, x) \leq b \vee n\}.$$

We call $< a, b >^n$ the annihilator of a relative to b around the element n or simply a relative n-annihilator. It is easy to see that for all $a, b \in S$, $< a, b >^n$ is always a convex subset containing n. In presence of distributivity, it can easily be seen that $< a, b >^n$ is an n-ideal. If $0 \in S$, then putting n =0, we have, $< a, b >^n = < a, b >$.

For two n-ideals A and B of a nearlattice S, $< A, B >$ denotes

$\{x \in S: m(a, n, x) \in B$ for all $a \in A\}$, when n is a medial element. In presence of distributivity, clearly $< A, B >$ is an n-ideal. Moreover, we can easily show that

$< a, b >^n = < <a>_n, <b>_n >$.

In this paper, we have given several characterizations of $< a, b>^n$. We have also given some characterizations of distributive and modular nearlattices in terms of relative n-annihilators.

.

## 1. Relative Annihilators around a central element of a Nearlattice.

We start with the following characterization of $< a, b >^n$.

**Theorem 1.1** *Let* S *be a nearlattice with a central element* n. *Then for all* a, b∈S, *the following conditions are equivalent.*

(i) $<a, b>^n$ *is an* n-*ideal*.

(ii) $< a \wedge n, b \wedge n >_d$ *is a filter and* $< a \vee n, b \vee n >$ *is an ideal*.

**Proof. (I)⇒(ii).** Suppose (i) holds. Let x, y∈$< a \vee n, b \vee n >$ and $x \vee y$ exists. Then $x \wedge (a \vee n) \leq (b \vee n)$. Thus $(x \wedge (a \vee n)) \vee n \leq (b \vee n)$, then by the neutrality of n, $(x \vee n) \wedge (a \vee n) \leq (b \vee n)$.

Also $m(x \vee n, n, a) = (x \vee n) \wedge (a \vee n) \leq b \vee n$. This implies $x \vee n \in < a, b >^n$. Similarly, $y \vee n \in < a, b >^n$. Since $< a, b >^n$ is an n-ideal,

so $x \vee y \vee n \in < a, b >^n$. This implies $m(x \vee y \vee n, n, a) \leq b \vee n$. That is,

$(x \vee y \vee n) \wedge (a \vee n) \leq b \vee n$ and so $(x \vee y) \wedge (a \vee n) \leq b \vee n$. Therefore,

$x \vee y \in < a \vee n, b \vee n >$.

Moreover, for $x \in < a \vee n, b \vee n >$ and $t \leq x$ (t∈S).

Obviously, $t \wedge (a \vee n) \leq b \vee n$, and so $t \in < a \vee n, b \vee n >$.

Hence $< a \vee n, b \vee n >$ is an ideal.

A dual proof of above shows that $< a \wedge n, b \wedge n >_d$ is a filter.

**(ii)⇒(i).** Suppose (ii) holds and x, y∈$< a, b >^n$.

Then $b \wedge n \leq (x \wedge a) \vee (x \wedge n) \vee (a \wedge n) \leq b \vee n$, and

$b \wedge n \leq (y \wedge a) \vee (y \wedge n) \vee (a \wedge n) \leq b \vee n$. So, $b \vee n \leq [(x \wedge a) \vee (x \wedge n) \vee (a \wedge n)] \wedge n = (x \wedge n) \vee (a \wedge n)$. This implies $x \wedge n \in < a \wedge n, b \wedge n >_d$. Similarly,

$y \wedge n \in < a \wedge n, b \wedge n >_d$. Since $< a \wedge n, b \wedge n >_d$ is a filter, so we have, $x \wedge y \wedge n \in < a \wedge n, b \wedge n >_d$. Thus, $(x \wedge y \wedge n) \vee (a \wedge n) \geq (b \wedge n)$.

But $m(x \wedge y \wedge n, n, a) = (x \wedge y \wedge n) \vee (a \wedge n) \geq (b \wedge n)$, and

so $x \wedge y \wedge n \in < a, b >^n$. Again, by neutrality of n, $(x \vee n) \wedge (a \vee n) = (x \wedge a) \vee n \leq (b \vee n)$. Similarly, $(y \vee n) \wedge (a \vee n) \leq (b \vee n)$.

Thus $((x \wedge y) \vee n) \wedge (a \vee n) \leq (b \vee n)$.

But $((x \wedge y) \vee n) \wedge (a \vee n) = m((x \wedge y) \vee n, n, a)$, as n is neutral.

There fore, $(x \wedge y) \vee n \in < a, b >^n$ and so by the convexity of $< a, b >^n$,

$x \wedge y \in < a, b >^n$.

A dual proof of above shows that $x \vee y \in < a, b >^n$. Clearly, $< a, b >^n$ contains n. Therefore, $< a, b >^n$ is an n –ideal. □

**Proposition 1.2** *Let* S *be a nearlattice with a central element* n. *Then for all* a, b∈S, *the following conditions hold.*

(i) $< a \vee n, b \vee n >$ *is an ideal if and only if* [n) *is a distributive subnearlattice of* S.

(ii) $< a \wedge n, b \wedge n >_d$ *is a filter if and only if* $(n]^d$ *is a distributive subnearlattice of* S.

**Proof.** Suppose for all a, b∈S, $< a \vee n, b \vee n >$ is an ideal. Thus for all p, q∈[n), $< p, q > \cap$ [n) is an ideal in the subnearlattice [n).Then by [1.1], [n) is distributive.

Conversely, suppose [n) is distributive. Let $x, y \in < a \vee n, b \vee n >$ and $x \vee y$ exists. Then

$x \wedge (a \vee n) \leq b \vee n$. Since n is neutral, so $(x \vee n) \wedge (a \vee n) =$

$[x \wedge (a \vee n)] \vee n \leq b \vee n$ implies that $x \vee n \in < a \vee n, b \vee n >$.

Similarly, $y \vee n \in < a \vee n, b \vee n >$. Then $(x \vee y) \wedge (a \vee n)$

$\leq (x \vee y \vee n) \wedge (a \vee n) = [(x \vee n) \wedge (a \vee n)] \vee [(y \vee n) \wedge (a \vee n)]$ as [n) is distributive.

$\leq (b \vee n)$.

Therefore, $x \vee y \in < a \vee n, b \vee n >$. Since $< a \vee n, b \vee n >$ has always the hereditary property, so $< a \vee n, b \vee n >$ is an ideal.

(ii) can be proved dually. □

By Theorem 1.1 and above result and using [8, theorem 1.5.2], we have the following result.

**Theorem 1.3** *Let* S *be a nearlattice with a central element* n. *Then for all* a, b∈S, $< a, b >^n$ is an n-ideal if and only if $P_n(S)$ is distributive nearlattice. □

Recall that a nearlattice S is distributive if for all x, y, z∈S, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ provided $y \vee z$ exists.[3] has given an alternative definition of distributivity of S. A nearlattice S is distributive if and only if for all t, x, y, z∈S, $t \wedge ((x \wedge y) \vee (x \wedge z)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$.

Similarly, by [4], a nearlattice S is modular if and only if for all t, x, y, z∈S with $z \leq x$, $x \wedge ((t \wedge y) \vee (t \wedge z)) = (x \wedge t \wedge y) \vee (t \wedge z)$.

Since for a sesquimedial element n, S is distributive if and only if $P_n(S)$ is distributive, we have the following Corollary, which is a generalization of [1, Theorem 1] and a result of [6]. This also generalizes a result of [7, theorem 3.1.3.].

**Corollary 1.4** *Suppose* S *is a nearlattice*. *Then for a central element* n∈S, $< a, b >^n$ *is an* n-*ideal for all* a, b∈S *if and only if* S *is distributive*. □

[1] gave a characterization of distributive lattices in terms of relative annihilators. Then [4] extended the result for nearlattices. [3] generalized the result for n-ideals in lattices. Following result gives a generalization of that result for n-ideals in nearlattices.

**Theorem 1.5** *Let* n *be a central element of a nearlattice* S. *Then the following conditions are equivalent*.

(i) S *is distributive*.

(ii) $< a \vee n, b \vee n >$ *is an ideal and* $< a \wedge n, b \wedge n >_d$ is *a filter whenever* $< b >_n \subseteq < a >_n$.

**Proof. (i)⇒(ii).** Suppose (i) holds. That is, S is distributive. Then by Corollary1.4, $< a, b >^n$ is an n-ideal for all a, b∈ S. Thus by Theorem 1.1, (ii) holds.

**(ii)⇒(i).** Suppose (ii) holds and let x, y, z∈[n) and $y \vee z$ exists.

Clearly, $(x \wedge y) \vee (x \wedge z) \leq x$. So, $< x, (x \wedge y) \vee (x \wedge z) >$ is an ideal as $< (x \wedge y) \vee (x \wedge z) >_n \subseteq < x >_n$. Since $x \wedge y \leq (x \wedge y) \vee (x \wedge z)$, so $y \in < x, (x \wedge y) \vee (x \wedge z) >$. Similarly, $z \in < x, (x \wedge y) \vee (x \wedge z) >$.

Hence $y \vee z \in < x, (x \wedge y) \vee (x \wedge z) >$ and so $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$. This implies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and so $[n)$ is distributive. Using the other part of (ii) we can similarly show that $(n]$ is also distributive. Thus by [8, theorem 1.5.2], $P_n(S)$ is distributive and so S is distributive. □

**Theorem 1.6** *Let* n *be a central element of a nearlattice* S. *Then the following conditions are equivalent.*

(i) $P_n(S)$ *is modular*.

(ii) *For* $a, b \in S$ *with* $< b >_n \subseteq < a >_n$, $x \in < b >_n$ and $y \in < a, b >^n$ *imply* $x \wedge y$, $x \vee y \in < a, b >^n$ *if* $x \vee y$ *exists in* S.

**Proof.** **(i)⇒(ii).** Suppose $P_n(S)$ is modular. Then by [8, theorem 1.5.2], $[n)$ and $(n]$ are modular. Here $< b >_n \subseteq < a >_n$, so $a \wedge n \leq b \wedge n \leq n \leq b \vee n \leq a \vee n$. Since $x \in < b >_n$, so $b \wedge n \leq x \leq b \vee n$.

Hence $a \wedge n \leq b \wedge n \leq x \wedge n \leq x \vee n \leq b \vee n \leq a \vee n$.

Now, $y \in < a, b >^n$ implies $m(y, n, a) \in < b >_n$.

Thus, $(y \wedge a) \vee (y \wedge n) \vee (a \wedge n) \leq b \vee n$, and so by the neutrality of n,

$((y \wedge a) \vee (y \wedge n) \vee (a \wedge n)) \vee n = (y \vee n) \wedge (a \vee n) \leq b \vee n$.

Thus, using the modularity of $[n)$ and the existence of $x \vee y$,

$m(x \vee y \vee n, n, a) = (x \vee y \vee n) \wedge (a \vee n)$

$= [(a \vee n) \wedge (y \vee n)] \vee (x \vee n)$ as $x \vee n \leq b \vee n \leq a \vee n$.

This implies $m(x \vee y \vee n, n, a) \leq b \vee n$ and so $x \vee y \vee n \in < a, b >^n$. Since n is neutral, so $a \wedge n \leq b \wedge n \leq x \wedge n$ implies that

$b \wedge n \leq (x \wedge n) \vee (y \wedge n) \vee (a \wedge n)$

$= ((x \vee y) \wedge n) \vee (a \wedge n)$

$= m((x \vee y) \wedge n, n, a)$

$\leq b \vee n$.

Therefore, $(x \vee y) \wedge n \in < a, b >^n$. Hence by convexity of $< a, b >^n$,

$x \vee y \in < a, b >^n$.

Again, using the modularity of (n], a dual proof of above shows that

$x \wedge y \in < a, b >^n$. Hence (ii) holds.

**(ii)⇒(i).** Suppose (ii) holds. Let $x, y, z \in [n)$ with $x \leq z$ and whenever $x \vee y$ exists. Then $x \vee (y \wedge z) \leq z$. This implies $< x \vee (y \wedge z) >_n \subseteq < z >_n$.

Now, $x \leq x \vee (y \wedge z)$ implies $x \in < x \vee (y \wedge z) >_n$.

Again, $y \wedge z \leq x \vee (y \wedge z)$ implies $m(y, n, z) = y \wedge z \in < x \vee (y \wedge z) >_n$.

Hence $y \in < z, x \vee (y \wedge z) >^n$. Thus by (ii), $x \vee y \in < z, x \vee (y \wedge z) >^n$. That is, $(x \vee y) \wedge z \leq x \vee (y \wedge z)$ and so $(x \vee y) \wedge z = x \vee (y \wedge z)$. Therefore, [n) is modular.

Similarly, using the condition (ii) we can easily show that (n] is also modular. Hence by [8, theorem 1.5.2], $P_n(S)$ is modular. □

We conclude this paper with the following characterization of minimal prime n-ideals belonging to an n-ideal. Since the proof of this is almost similar to [8, theorem 2.1.4], we omit the proof.

**Theorem 1.7** *Let* S *be a distributive nearlattice and* P *be a prime* n-*ideal of* S *belonging to an* n-*ideal* J. *Then the following conditions are equivalent.*

(i) P *is minimal prime* n-*ideal belonging to* J.

(ii) $x \in P$ *implies* $< < x >_n, J > \not\subseteq P$. □

## REFERENCES


1. M. Mandelker, *Relative annihilators in lattices*, Duke Math. J. 40, 377-386(1970).

2. J. Varlet, *Relative annihilators in semi-lattices*, Bull. Austral. Math. Soc. 9, 169-185(1973).

3. A.S.A. Noor and M. A. Ali, *Relative annihilators around a neutral element of a lattice*, Rajshahi University Studies (Part-B), 28,141-146(2000).

4. A.S.A. Noor and A.K.M.S. Islam, *Relative annihilators in nearlattices*, Rajshahi University Studies (Part-B), J. Sci. 25, 117-120(1997).

5. W.H. Cornish, *Normal lattices*, J. Austral. Math. Soc. 14, 200-215(1972).

6. A.S.A. Noor and M.A. Ali, *Minimal Prime* n-*ideals of a lattice*, Journal of Science, North Bengal University Review, India, 9(1), 30-36(1998).

7. M.A. Ali, *A study on finitely generated* n-*ideals of a lattice*, Ph.D. Thesis, Rajshshi University, Rajshahi(1998).

8. Raihan M.S., On Principal n-Ideals of a Distributive Nearlattice, Ph.D. Thesis, Rajshahi University, Rajshahi(2006).