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Characterizations of m-Normal Nearlattices in terms of Principal n-Ideals

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Abstract

A convex subnearlattice of a nearlattice S containing a fixed element $n \in S$ is called an n-ideal. The n-ideal generated by a single element is called a principal n-ideal. The set of finitely generated principal n-ideals is denoted by $P_n(S)$, which is a nearlattice. A distributive nearlattice S with 0 is called m-normal if its every prime ideal contains at most m number of minimal prime ideals. In this paper, we include several characterizations of those $P_n(S)$ which form m-normal nearlattices. We also show that $P_n(S)$ is m-normal if and only if for any m+1 distinct minimal prime n-ideals $P_o, P_1, \ldots, P_m$ of S, $P_o \vee \ldots \vee P_m = S$.

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Introduction

Lee in [9], also see Lakser [7], has determined the lattice of all equational subclasses of the class of all pseudo-complemented distributive lattices. They are given by

$B_{-1} \subset B_o \subset \ldots \subset B_m \subset \ldots \subset B\omega$, where all the inclusions are proper and $B\omega$ is the class of all pseudo-complemented distributive lattices, B_{-1} consists of all one element algebra, B_o is the variety of Boolean algebras while B_m, for $-1 \leq m < \omega$ consists of all algebras satisfying the equation

$$(x_1 \wedge x_2 \wedge \ldots \wedge x_m)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge x_2 \wedge \ldots \wedge x_{i-1} \wedge x_i^* \wedge x_{i+1} \wedge \ldots \wedge x_m)^* = 1$$

where x^* denotes the pseudo-complemente of x. Thus B_1 consists of all Stone algebras.

Davey [4] has independently given several characterizations of (sectionally) B_m and relatively B_m-lattices. On the other hand Cornish in [3] has studied distributive lattices (without pseudo-complementation) analogues to B_m-lattices and relatively B_m-lattices.

A distributive nearlattice S with 0 is called m-*normal* if each prime ideal of L contains at most m-minimal prime ideals. For a fixed element $n \in S$, a convex subnearlattice containing n is called an n-ideal. An n-ideal generated by a finite number of elements $a_1, a_2, \ldots, a_n$ is called a *finitely generated* n-*ideal*, denoted by $< a_1, a_2, \ldots, a_n >_n$. The set of all finitely generated n-ideals is a nearlattice denoted by $F_n(S)$. An n-ideal generated by a single element is called a principal n-ideal is denoted by $P_n(S)$.

In this paper we include several characterizations of those $P_n(S)$ which form m-normal nearlattices. We show that $P_n(S)$ is m-normal if and only if for any m+1 distinct minimal prime n-ideals $P_o, P_1, \ldots, P_m$ of S, $P_o \vee \ldots \vee P_m = S$.

We start the paper with the following result on n-ideals due to Latif and Noor [8].

Lemma 1.1 For a central element $n \in S$, $P_n(S) \cong (n]^d \times [n)$.

Following result is also essential for the development of this paper, which is due to Ali [1,Theorem 1.1.12].

Lemma 1.2 *Let* S *be a distributive near-lattice with an upper element* n *and let* I , J *be two* n-*ideals of* S. *Then for any* $x \in I \vee J$, $x \vee n = i \vee j$ *and* $x \wedge n = i' \wedge j'$ *for some* $i, i' \in I$, $j, j' \in J$ *with* $i, j \geq n$ *and* $i', j' \leq n$.

Now we include the following result which is due to Noor and Ali [10] and this is a generalization of [2, Lemma 3.6].

A prime n-ideal P is said to be a *minimal prime* n-*ideal* belonging to n-ideal I if

(i) $I \subseteq P$ and

(ii) There exists no prime n-ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$.

A prime n-ideal P of a nearlattice S is called a *minimal prime* n-*ideal* if there exists no prime n-ideal Q such that $Q \neq P$ and $Q \subseteq P$. Thus a minimal prime n-ideal is a minimal prime n-ideal belonging to $\{n\}$.

Following lemma will be needed for further development of this paper. This is [3, Lemma 3.6] and is easy to prove. So we omit the proof.

The following result is [4, Lemma 2.2] which also follows from the corresponding result for commutative semi-groups due to Kist [6].

Lemma 1.3 *Let* M *be a prime ideal containing an ideal* J *in a distributive medial nearlattice*. *Then* M *is a minimal prime ideal belonging to* J *if and only if for all* $x \in M$, *there exists* $x' \notin M$ *such that* $x \wedge x' \in J$. □

Now we generalize this result for n-ideals.

Lemma 1.4 *Let* n *be a medial element and* M *be a prime* n-*ideal containing an* n-*ideal* J. *Then* M *is a minimal prime* n-*ideal belonging to* J *if and only if for all* $x \in M$ *there exists* $x' \notin M$ *such that* $m(x, n, x') \in J$.

Proof. Let M be a minimal prime n-ideal belonging to J and $x \in M$. Then by [11], $<< a >_n, J > \not\subset M$. So there exists x' with $m(x, n, x') \in J$ such that $x' \notin M$.

Conversely, suppose $x \in M$, then there exists $x' \notin M$ such that $m(x, n, x') \in J$. This implies $x' \notin M$, but $x' \in << x >_n, J >$, that is $<< x >_n, J > \not\subset M$. Hence by [10], M is a prime n-ideal belonging to J. □

Davey in [4, Corollary 2.3] used the following result in proving several equivalent conditions on B_m-lattices. On the other hand, Cornish in [3] has used this result in studing n-normal lattices.

Proposition 1.5 *Let* $M_o, \ldots, M_n$ *be* n+1 *distinct minimal prime ideals of a distributive nearlattice* S. *Then there exists* $a_o, a_1, \ldots, a_n \in S$ *such that* $a_i \wedge a_j \in J$ $(i \neq j)$ *and* $a_j \notin M_j$, $j = 0, 1, \ldots, n$. □

Now we generalize the above result in terms of n-ideals.

Proposition 1.6 Let S be a distributive nearlattice and $n \in S$ is medial. Suppose $M_o, .., M_m$ be m+1 distinct minimal prime n-ideals containing n-ideal J. Then there exists $a_o, a_1, \ldots, a_n \in S$ such that $m(a_i, n, a_j) \in J$ ($i \neq j$) and $a_j \notin M_j$ ($j = 0, 1, \ldots, m$).

Proof. Let n =1. Let $x_o \in M_1 - M_o$ and $x_1 \in M_o - M_1$. Then by Lemma 1.3, there exists $x_1' \notin M_o$ such that $m(x_o, n, x_1') \in J$. Hence $a_1 = x_1$, $a_o = m(x_o, n, x_1')$ are the required elements.

Observe that $m(a_o, n, a_1) = m(m(x_o, n, x_1'), n, x_1)$

$$= (x_o \wedge x_1 \wedge x_1') \vee (x_o \wedge n) \vee (x_1 \wedge n) \vee (x_1' \wedge n)$$

$$= (x_o \wedge m(x_1, n, x_1')) \vee (x_o \wedge n) \vee (m(x_1, n, x_1') \wedge n)$$

$$= m(x_o, n, m(x_1, n, x_1'))$$

Now, $m(x_1, n, x_1') \wedge n \leq m(x_o, n, m(x_1, n, x_1'))$

$$\leq \quad m(x_1, n, x_1') \vee n$$

and $m(x_1, n, x_1') \in J$, so by convexity $m(a_o, n, a_1) \in J$.

Assume that, the result is true for n = m-1, and let $M_o, \ldots, M_m$ be m+1 distinct minimal prime n-ideals. Let b_j ($j = 0, 1, \ldots, m-1$) satisfy

$m(b_i, n, b_j) \in J$ ($i \neq j$) and $b_j \notin M_j$. Now choose $b_m \in M_m - \bigcup_{j=0}^{m-1} M_j$ and by Lemma 1.4, let $b_{m'}$ satisfy $b_{m'} \notin M_m$ and $m(b_m, n, b_{m'}) \in J$. Clearly,

$a_j = m(b_j, n, b_m)$ ($j = 0, \ldots, m-1$) and $a_m = b_{m'}$, establish the result. □

Let J be an n-ideal of a distributive lattice L. A set of elements $x_o, \ldots, x_n \in L$ is said to be *pairwise* in J if $m(x_i, n, x_j) = n$ for all $i \neq j$.

The next result is [3, Lemma 2.3] which was suggested by Hindman in

[5, Theorem 1.8].

Lemma 1.7 *Let* J *be an ideal in a distributive nearlattice* S. *For a given positive integer* $n \geq 2$, *the following conditions are equivalent.*

(i) *For any* $x_1, \ldots, x_n \in S$ *which are 'pairwise in* J' *that is* $x_i \wedge x_j \in J$ *for any* $i \neq j$, *there exists* k *such that* $x_k \in J$.

(ii) *For any ideals* $J_1,\ldots,J_n$ *in* S *such that* $J_i \cap J_j \subseteq J$ *for any* $i \neq j$,

there exists k *such that* $J_k \subseteq J$.

(iii) J *is the intersection of at most* n-1 *distinct prime ideals*. □

Our next result is a generalization of above result. This result will be needed in proving the next theorem which is the main result of this section. In fact, the following lemma is very useful in studing those $P_n(S)$ which are m-normal.

Lemma 1.8 *Let* J *be an* n-*ideal in a distributive nearlattice* S *and* $n \in S$ *is medial. For a given positive integer* $m \geq 2$, *the following conditions are equivalent.*

(i) *For any* $x_1,\ldots, x_n \in S$ *with* $m(x_i, n, x_j) \in J$ (*that is, they are pairwise in* J) *for any* $i \neq j$, *there exists* k *such that* $x_k \in J$.

(ii) *For any* n- *ideals* $J_1,...,J_m$ *in* S *such that* $J_i \cap J_j \subseteq J$ *for any* $i \neq j$, *there exists* k *such that* $J_k \subseteq J$.

(iii) J *is the intersection of at most* m-1 *distinct prime* n- *ideals*.

Proof. (i) and (ii) are easily seen to be equivalent.

(iii)⇒(i). Suppose $P_1, P_2,\ldots,P_k$ are k $(1 \leq k \leq m\text{-}1)$ distinct prime n-ideals such that $J = P_1 \cap P_2 \cap \ldots \cap P_k$. Let $x_1, x_2,\ldots, x_m$ S be such that $m(x_i, n, x_j) \in J$ for all $i \neq j$. Suppose no element x_i is a member of J.Then for each r $(1 \leq r \leq k)$ there is at most one i $(1 \leq i \leq m)$ such that $x_i \in P_r$. Since $k < m$, there is some i such that $x_i \in P_1 \cap P_2 \cap \ldots \cap P_k$.

(i)⇒(iii). Suppose (i) holds for m = 2, then it implies that J is a prime n-ideal. Then (iii) is trivially true. Thus we may assume that there is a largest integer t with $2 \leq t < m$ such that the condition (i) does not hold for J (consequently condition (i) holds for t+1, t+2,…, m). Then for some $2 \leq t < m$ we may suppose that there exist elements $a_1, a_2,\ldots, a_t \in L$ such that $m(a_i, n, a_j) \in J$ for $i \neq j$, $i = 1, 2,\ldots, t$, $j = 1, 2,\ldots, t$, yet $a_1, a_2,\ldots, a_t \notin J$.

As S is a distributive lattice, $<< a_i >_n, J >$ is an n-ideal for any $i \in \{1, 2,\ldots, t\}$.

Each $<< a_i >_n, J >$ is in fact a prime n-ideal.

Firstly $<<a_i>_n, J> \neq S$, since $a_i \notin J$. Secondly, suppose that b and c are in S and $m(b, n, c) \in <<a_i>_n, J>$. Consider the set of t+1 elements $\{a_1, a_2, \ldots, a_{i-1}, m(b, n, a_i), m(c, n, a_i), a_{i+1}, \ldots, a_t\}$. This set is pairwise in J and

so, either $m(b, n, a_i) \in J$ or $m(c, n, a_i) \in J$. Since condition (i) holds for t+1.

That is, $b \in <<a_i>_n, J>$ or $c \in <<a_i>_n, J>$ and so $<<a_i>_n, J>$ is prime.

Clearly, $J \subseteq \bigcap_{1\le i\le t} <<a_i>_n, J>$. If $w \in \bigcap_{1\le i\le t} <<a_i>_n, J>$. Then $w, a_1, a_2, \ldots, a_t$ are pairwise in j and so $w \in J$. Hence $J = \bigcap_{1\le i\le t} <<a_i>_n, J>$ is the intersection of $t < m$ prime n-ideals. □

An ideal $J \neq S$ satisfying the equivalent conditions of Lemma 1.7. is called an m-*prime ideal*. Similarly, an n-ideal $J \neq S$ satisfying the equivalent conditions of Lemma 1.8. is called an m-*prime* n-*ideal*.

For a, b $\in$S, $<a, b> = \{x \in S: x \wedge a \le b\}$ is known as annihilator of a relative to b or simply a relative annihilator. In presence of distributivity, it is easy to show that each relative annihilator is an ideal. Again for $a, b \in L$, where L is a lattice, we define

$<a, b>_d = \{x \in L: x \vee a \ge b\}$ is a relative dual annihilator. In presence of distributivity of L, $<a, b>_d$ is a dual ideal (filter).

For $a, b \in S$ and an upper element $n \in S$,

we define, $<a, b>^n = \{x \in S: m(a, n, x) \in <b>_n\}$

$$= \{x \in S: b \wedge n \le m(a, n, x) \le b \vee n\}.$$

We call $<a, b>^n$ the annihilator of a relative to b around the element n or simply a relative n-annihilator. It is easy to see that for all $a, b \in S$, $<a, b>^n$ is always a convex subset containing n. In presence of distributivity, it can easily be seen that $<a, b>^n$ is an n-ideal. If $0 \in S$, then putting n =0, we have, $<a, b>^n = <a, b>$.

For two n-ideals A and B of a nearlattice S, $< A, B >$ denotes $\{x \in S: m(a, n, x) \in B$ for all $a \in A\}$, when n is a medial element. In presence of distributivity, clearly $< A, B >$ is an n-ideal.

Now we generalize a result of Davey in [4, Proposition 3.1.].

Theorem 1.9 *Let* J *be an* n*-ideal of a distributive nearlattice* S *and* n *be a central element of* S. *Then the following conditions are equivalent.*

(i) *For any* $m+1$ *distinct prime* n*-ideals* $P_o, P_1, \ldots, P_m$ *belonging to* J, $P_o \vee P_1 \vee \ldots \vee P_m = S$.

(ii) *Every prime* n*-ideal containing* J *contains at most m distinct minimal prime* n*-ideals belonging to* J.

(iii) *If* $a_o, a_1, \ldots, a_m \in S$ *with* $m(a_i, n, a_j) \in J$ $(i \neq j)$ *then* $\bigvee_j << a_j >_n, J > = S$.

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii). Assume $a_o, a_1, \ldots, a_m \in S$ with $m(a_i, n, a_j) \in J$ and $\bigvee_j << a_j >_n, J > \neq S$. It follows that $a_j \notin J$, for all j. Then by [8], there exists a prime n-ideal P such that $\bigvee_j << a_j >_n, J > \subseteq P$. But by [11], we know that P is either a prime ideal or a prime filter.

Suppose P is a prime ideal. For each j, let $F_j = \{x \wedge y: x \geq a_j, x, y \geq n, y \notin P\}$.

Let $x_1 \wedge y_1, x_2 \wedge y_2 \in F_j$.

Then $(x_1 \wedge y_1) \wedge (x_2 \wedge y_2) = (x_1 \wedge x_2) \wedge (y_1 \wedge y_2)$.

Now, $x_1 \wedge x_2 \geq a_j$ and $y_1 \wedge y_2 = m(y_1, n, y_2)$. So $t \geq x \wedge y$ implies $t = (t \vee x) \wedge (t \vee y)$. Since $y \notin P$, so $t \vee y \notin P$. Hence $t \in F_j$, and so F_j is a dual ideal.

We now show that $F_j \cap J = \phi$, for all $j = 0, 1, 2, \ldots, m$. If not let $b \in F_j \cap J$, then $b = x \wedge y$, $x \geq a_j$, $x, y \geq n$, $y \notin P$. Hence $m(a_j, n, y) = (a_j \wedge n) \vee n \vee (a_j \wedge y) = (a_j \wedge y) \vee n = (a_j \vee n) \wedge (y \vee n)$. But $(a_j \vee n) \wedge (y \vee n) \in F_j$ and

$n \le (a_j \wedge y) \vee n \le b$ implies $m(a_j, n, y) \in J$. Therefore, $m(a_j, n, y) \in F_j \cap J$.

Again, $m(a_j, n, y) \in J$ with $y \notin P$ implies $<< a_j >_n, J> \not\subset P$, which is a contradiction. Hence $F_j \cap J = \phi$ for all j. For each j, let P_j be a minimal prime n-ideal belonging to J and $F_j \cap P_j = \phi$. Let $y \in P_j$. If $y \notin P$, then $y \vee n \notin P$.

Then $m(a_j, n, y \vee n) = (a_j \vee n) \wedge (y \vee n) \in F_j$.

But $m(a_j, n, y \vee n) \in < y \vee n >_n \subseteq < y >_n \subseteq P_j$, which is a contradiction.

So $y \in P$. Therefore $P_j \subseteq P$, and $a_j \notin P_j$. For if $a_j \in P_j$, then $a_j \vee n \in P_j$.

Now, $a_j \vee n = (a_j \vee n) \wedge (a_j \vee n \vee y) \in F_j$ for any $y \notin P$. This implies $P_j \cap F_j \neq \emptyset$, which is a contradiction. So, $a_j \notin P_j$. But $m(a_i, n, a_j) \in J \subseteq P_j$ $(i \neq j)$ which implies $a_i \in P_j$ $(i \neq j)$ as P_j is prime. It follows that P_j form a set of m+1 distinct minimal prime n-ideals belonging to J and contained in P. This contradicts (ii).

Therefore, $\bigvee_j << a_j >_n, J> = S$.

Similarly, if P is filter, then a dual proof of above also shows that

$\bigvee_j << a_j >_n, J> = S$, and hence (iii) holds.

(iii)⇒(i). Let $P_o, P_1, \ldots, P_m$ be m+1 distinct minimal prime n-ideals belonging to J. Then by Proposition 1.6, there exists

$a_o, a_1, \ldots, a_m \in S$ such that $m(a_i, n, a_j) \in J$ $(i \neq j)$ and $a_j \notin P_j$. This implies

$<< a_j >_n, J> \subseteq P_j$ for all j. Then by (iii),

$<< a_o >_n, J> \vee << a_1 >_n, J> \vee \ldots \vee << a_m >_n, J> \subseteq P_o \vee P_1 \vee \ldots \vee P_m$, which implies $P_o \vee P_1 \vee \ldots \vee P_m = S$. □

For a prime n-ideal P of S, $n(P) = \{x \in S: m(x, n, y) = n$ for some $y \in S-P\}$. Clearly, n(P) is an n-ideal and $n(P) \subseteq P$. Our next result is a nice extension of above result in terms of n-ideals.

Theorem 1.10. *Let* S *be a distributive nearlattice with a central element* n. *Then the following conditions are equivalent.*

(i) *For any* m+1 *distinct minimal prime* n-*ideals* $P_o, P_1, \ldots, P_m$,

$P_o \vee P_1 \vee \ldots. \vee P_m = S$.

(ii) *Every prime* n- *ideal contains at most* m *minimal prime* n- *ideals*.

(iii) *For any* $a_o, a_1, \ldots., a_m \in S$ *with* $m(a_i, n, a_j) = n$ *for* $(i \neq j)$,

$i = 0, 1,2,---,m,\ j = 0, 1,2,---,m,\ <a_o>_n^* \vee <a_1>_n^* \vee \ldots \vee <a_m>_n^* = S$.

(iv) *For each prime* n- *ideal* P, n(P) *is an* m+1- *prime* n-*ideal*.

Proof. **(i)⇒(ii)**, **(ii)⇒(iii)** and **(iii)⇒(i)** easily hold by Theorem 1.9, replacing J by {n}.To complete the proof we need to show that **(iv)⇒(iii)** and **(ii)⇒(iv)**.

(iv)⇒(iii). Suppose (iv) holds and $x_o, x_1, \ldots, x_m$ are m+1 elements of S such that $m(x_i, n, x_j) = n$ for $(i \neq j)$. Suppose that $<x_o>_n^* \vee \ldots \vee <x_m>_n^* \neq S$. Then by Stone's separation theorem in [9], there is a prime n-ideal P such that $<x_o>_n^* \vee \ldots \vee <x_m>_n^* \subseteq P$. Hence $x_o, x_1, \ldots, x_m \in S - n(P)$. This contradicts (iv) by Lemma 1.8, since $m(x_i, n, x_j) = n \in n(P)$ for all $i \neq j$. Thus (iii) holds.

(ii)⇒(iv). This follows immediately from Lemma 1.8 □

Proposition 1.11 *Let* S *be a distributive medial nearlattice and* n∈S *is a central element. If the equivalent conditions of Theorem* 1.10 *hold, then for any* m+1 *elements* $x_o, x_1, \ldots, x_m$; $(<x_o>_n \cap <x_1>_n \cap \ldots \cap <x_m>_n)^* =$

$$\bigvee_{0 \le i \le n} (<x_o>_n \cap <x_1>_n \cap \ldots \cap <x_{i-1}>_n \cap <x_{i+1}>_n \cap \ldots \cap <x_m>_n)^*.$$

Proof. Let $<b_i>_n = <x_o>_n \cap <x_1>_n \cap \ldots \cap <x_{i-1}>_n \cap <x_{i+1}>_n \cap \ldots \cap <x_m>_n$ for each $0 \le i \le m$. Suppose $x \in (<x_o>_n \cap <x_1>_n \cap \ldots \cap <x_m>_n)^*$.

Then $<x>_n \cap <x_o>_n \cap <x_1>_n \cap \ldots \cap <x_m>_n = \{n\}$. For all $i \neq j$,

$(<x>_n \cap <b_i>_n) \cap (<x>_n \cap <b_j>_n) = \{n\}$.

So $(<x>_n \cap <b_o>_n)^* \vee \ldots \vee (<x>_n \cap <b_m>_n)^* = S$.

Thus $x \in (<x>_n \cap <b_o>_n)^* \vee \ldots \vee (<x>_n \cap <b_m>_n)^*$. Hence by Lemma 1.2,

$x \vee n = a_o \vee \ldots. \vee a_m$ where $a_i \in (<x>_n \cap <b_i>_n)^*$ and $a_i \ge n$ for

$i = 0,1,\ldots,m$. Then $x \vee n = (a_o \wedge (x \vee n)) \vee \ldots \vee (a_m \wedge (x \vee n))$.

Now $a_i \in (< x >_n \cap < b_i >_n)^*$ implies $< a_i >_n \cap < x >_n \cap < b_i >_n = \{n\}$. Then by a routine calculation we find that $(a_i \wedge x \wedge b_i) \vee n = n$

Thus $< a_i \wedge (x \vee n) >_n \cap < b_i >_n = [n, (a_i \wedge x \wedge b_i) \vee n] = \{n\}$ implies that

$a_i \wedge (x \vee n) \in < b_i >_n^*$ and so $x \vee n \in < b_o >_n^* \vee < b_1 >_n^* \vee \ldots \vee < b_m >_n^*$. By a dual proof of above and using Theorem 1.3.7, we can easily show that

$x \wedge n \in < b_o >_n^* \vee < b_1 >_n^* \vee \ldots \vee < b_m >_n^*$.

Thus by convexity, $x \in < b_o >_n^* \vee < b_1 >_n^* \vee \ldots \vee < b_m >_n^*$.

This proves that L.H.S. $\subseteq$ R.H.S. The reverse inclusion is clear. □

Theorem 1.12 *Let* S *be a distributive nearlattice and* $n \in S$ *is central. Then the following conditions are equivalent.*

(i) $P_n(S)$ *is* m-*normal*.

(ii) *Every prime* n-*ideal contains at most* m *minimal prime* n-*ideals*.

(iii) *For any* m+1 *distinct minimal prime* n-*ideals* $P_o, \ldots, P_m$;

$P_o \vee \ldots \vee P_m = S$.

(iv) *If* $m(a_i, n, a_j) = n$, *this implies* $< a_o >_n^* \vee \ldots \vee < a_m >_n^* = S$.

(v) *For each prime* n-*ideal* P, n(P) *is an* m+1 *prime* n-*ideal*.

Proof. (i)⇒(ii). Let $P_n(S)$ be m-normal, since n is central, $P_n(S) \cong (n]^d \times [n)$, so both $(n]^d$ and $[n)$ are m-normal. Suppose P is any prime n-ideal of S. Then by [10], either $P \supseteq (n]$ or $P \supseteq [n)$. Without loss of generality, suppose $P \supseteq [n)$. Then by [10], P is prime ideal of S. Hence by [2, Lemma 3.4], $P_1 = P \cap [n)$ is a prime ideal of [n). Since [n) is m-normal, so by [3] P_1 contains at most m minimal prime ideals $R_1, R_2, \ldots, R_m$ of [n). Therefore, P contains at most m minimal prime ideals $T_1, T_2, \ldots, T_m$ of S where

$R_1 = T_1 \cap [n)$, $R_2 = T_2 \cap [n), \ldots, R_m = T_m \cap [n)$. Since $n \in R_1, \ldots, R_m$, $n \in T_1, \ldots, T_m$, hence $T_1, \ldots, T_m$ are minimal prime n-ideals of S. Thus (ii) holds.

(ii)⇒(i). Suppose (ii) holds. Let P_1 be a prime ideal in [n). Then by

[2, Lemma 3.4], $P_1 = P \cap [n)$ for some prime ideal P of S. Since

$n \in P_1 \subseteq P$, so P is prime n-ideal. Therefore, P contains at most m minimal prime n-ideals $R_1, \ldots, R_m$ of S. Thus by [2, Lemma 3.4], P_1 contains at most m minimal prime ideals $T_1 = R_1 \cap [n)$, $T_2 = R_2 \cap [n), \ldots\ldots$, $T_m = R_m \cap [n)$ of [n). Hence by Theorem 1.10, [n) is m-normal. Similarly, we can prove that $(n]^d$ is also m-normal. Thus by Lemma 1.1, $P_n(S)$ is m-normal.

(ii) ⇔ **(iii)** has already been proved in Theorem 1.10 □

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