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Analysis of Mixed Effects Model for Randomized Block Designs with Both-sided Neighbor-Effects

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Abstract

In this study we confine our attention to Randomized Block Designs (RBDs) with random block effects in the presence of both-sided neighbor-effects and focus our attention to (i) estimation of variance components, and (ii) estimation of treatment contrasts as well as both left-sided and right sided neighbor-effects contrasts. We start with a simple set-up of 4 blocks and 4 treatments in an RBD. This we do by concurrently studying three distinct RBDs with the same parameters. However, the allocations of Left Neighbor (LN)- and Right Neighbor (RN)-Effects are assumed to be different across the RBDs.

Keywords and Phrases: Random block effects, variance components, left- and right-neighbor effects, treatment contrasts.

AMS Classification: 62K10.

1. Introduction

Random effects models are thoroughly studied and reported in text books. The importance of the present study is two-fold: we introduce both left-sided and right-sided neighbor effects in a block design with random block effects and examine the questions of estimability of the two variance components, besides that of the treatment contrasts and neighbor-effects contrasts. For this we will examine the collective status of 3 RBDs, each with b = v = 4. We list the RBDs below.

Table 1: Three RBDs in 4 random blocks and 4 treatments

RBD - I

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

RBD - II

1	2	3	4
1	2	4	3
2	1	3	4
2	1	4	3

RBD - III

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

The model stipulates iid random block effects with mean zero and variance σ_b^2 while the error variance is denoted by σ_e^2. The treatment effects are denoted by τ_1, τ_2, τ_3, τ_4. The left-neighbor-effects are denoted by LN1, LN2, LN3, LN4 and the right-neighbor effects are denoted by RN1, RN2, RN3, RN4. For completeness, we describe the model below where the notations have their usual significance. Below y_{ij} stands for the output in the plot in block i involving treatment j. The indicator functions determine the LN- and RN-effects associated with the designated plot(s).

$$y_{ij} = \mu + b_i + \tau_j + \sum_k I(LN(i,j;k)) + \sum_k I(RN(i,j;k)) + e_{ij}$$

We now aim at the data analysis. This is accomplished in the subsequent sections. It is not straightforward, as a matter of fact. There are many issues to be settled one by one. These relate to estimation of (i) treatment contrasts, (ii) LN-effects contrasts, (iii) RN-effects contrasts, and (iv) the two variance components.

We refer to our previous works in models with neighbor effects Sapam et al (2019a, 2019b), Sapam and Sinha (2020, 2021a, 2021b). Further, we refer to Bhowmik et al (2019) for a study on neighbor designs with random block effects. Also see Jaggi et al (2018), Azais et al (1993) in this context. Finally, we also refer to Searle (1971) for an introduction to random effects models and variance components estimation.

2. Study of the Nature of Error Functions

In each RBD with random block effects, we have a total of 15 df and in the absence of Neighbor-Effects (N-Effects), we have only 3 treatment contrasts and the rest are all errors. First we classify the errors and then examine their status in the presence of N-Effects. Below we confine to RBD-I.

Table 2 [RBD-I.1]: Status of Three Error Functions based on Within Block Contrasts of Block 1 and Block 2 - in the presence of Neighbor-Effects for the RBD-I

Sl No.	Error functions	Observational Contrast Based on Block 1 - Block 2	L-N Effects in model expectation	R-N Effects in model expectation
1	Error 1	[y(1,1)−y(1,2)]−[y(2,4)−y(2,1)]	(LN4 − LN1) − (LN4 − LN1) = 0	(RN2 − RN3) − (RN2 − RN3) = 0
2	Error 2	[y(1,1)−y(1,3)] −[y(2,4)−y(2,2)]	(LN4 − LN2) − (LN4 − LN2) = 0	(RN2 − RN4) − (RN2 − RN4) = 0
3	Error 3	[y(1,1)−y(1,4)]−[y(2,4)−y(2,3)]	(LN4 − LN3) − (LN4 −LN3) = 0	(RN2 − RN1) − (RN2 − RN1) = 0

Table 3 [RBD-I.2]: Status of Three Error Functions based on Within Block Contrasts of Block 1 and Block 3 - in the presence of Neighbor-Effects for the RBD-I

Sl. No	Error functions	Observational Contrast Based on Block 1 - Block 3	L-N Effects in model expectation	R-N Effects in model expectation
1	Error 4	[y(1,1) −y(1,2)] − [y(3,3) −y(3,4)]	(LN4 −LN1) – (LN4 − LN1) = 0	(RN2−RN3) −(RN2−RN3) = 0
2	Error 5	[y(1,1) −y(1,3)] − [y(3,3) −y(3,1)]	(LN4 − LN2) − (LN4 – LN2) =0	(RN2 − RN4) − (RN2 – RN4) = 0
3	Error 6	[y(1,1) −y(1,4)] − [y(3,3) −y(3,2)]	(LN4−LN3) − (LN4 – LN3) = 0	(RN2−RN1) − (RN2 – RN1) = 0

Table 4 [RBD-I.3]: Status of Three Error Functions based on Within Block Contrasts of Block 1 and Block 4 - in the presence of Neighbor-Effects for the RBD-I

Sl. No	Error functions	Observational Contrast Based on Block 1 - Block 4	L-N Effects in model expectation	R-N Effects in model expectation
1	Error 7	[y(1,1)−y(1,2)] − [y(4,2)−y(4,3)]	(LN4 − LN1) − (LN4 − LN1) = 0	(RN2 − RN3) − (RN2−RN3) =0
2	Error 8	[y(1,1)−y(1,3)] −[y(4,2)−y(4,4)]	(LN4 − LN2) − (LN4 − LN2) = 0	(RN2 − RN4) −(RN2 −RN4) =0
3	Error 9	[y(1,1)−y(1,4)]−[y(4,2)−y(4,1)]	(LN4 − LN3) − (LN4 − LN3) = 0	(RN2 − RN1) − (RN2 −RN1)= 0

Remark 1. Note that Errors 1 to 9 listed above turn out to be pure errors relating to estimation of error variance σ_e^2. This is a very encouraging result from the point of view of estimation and testing. In table 5, we confine to the last set of three error functions wrt RBD-I. These are based on contrasts involving the four Block Totals. Recall that in a fixed-effects RBD, there are 3 error df due to blocks. When the block effects are treated as random, these 3 df are meant to provide estimate of $\sigma_b^2 + {\sigma_e^2}/{4}$. This is so when the Neighbor- Effects are absent. Therefore, we need to check the status of these block total contrasts in the presence of Neighbor- Effects. This is carried out in Table 5 below. Note that the treatment effects contrasts are already orthogonal to the block effects contrasts.

Table 5 [RBD-I]: Status of Three Block Total Contrasts in the presence of Neighbor - Effects for the RBD-I

Sl. No	Observational Contrast Based on Block 1 Total - Block j	Total L-N Effects in model expectation	R-N Effects in model expectation
j = 2	[y(1,1) + y(1,2) + y(1,3) + y(1,4)] − [y(2,1) + y(2,2) + y(2,3) + y(2,4)]	0	0
j= 3	[y(1,1) + y(1,2) + y(1,3) + y(1,4)] − [y(3,1) + y(3,2) + y(3,3) + y(3,4)]	0	0
j = 4	[y(1,1) + y(1,2) + y(1,3) + y(1,4)] − [y(4,2) + y(4,3) + y(4,4) + y(4,1)]	0	0

Remark 2. This is also an encouraging result. We can provide estimates for σ_e^2 and σ_b^2 in a routine manner. Usual Error Mean Square in the RBD-I will provide estimate of σ_e^2 while usual Block Mean Square in the RBD-I will provide estimate of $\sigma_b^2 + {\sigma_e^2}/{4}$. It is known that non-negative variance estimation problem cannot be avoided. For details about available methods to tackle this issue, we refer to Searle (1971).

Remark 3. Unfortunately, as will be seen later, the three treatment contrasts, computed on the basis of 4 treatment totals of the RBD-I involve, respectively, (i) LN4−LN1 +RN2−RN3, (ii) LN4 − LN2 + RN2 − RN4 and (iii) LN4 − LN3 + RN2 − RN1, in the presence of N-Effects. Therefore, in RBD-I, treatment contrasts are not estimable. Now we may proceed to carry out similar data analysis for RBD-II.

Table 6 [RBD-II.1]: Status of Three Error Functions based on Within Block Contrasts of Block 1 and Block 2 - in the presence of Neighbor-Effects for the RBD-II

Sl. No	Error functions	Observational Contrast Based on Block 1 − Block 2	L-N Effects in model expectation	R-N Effects in model expectation
1	Error 1	[y(1,1)−y(1,2)]−[y(2,1)−y(2,2)]	[(LN4 − LN1)] − [(LN3 −LN1)] = (LN4−LN3)	[(RN2 − RN3)] − [(RN2 −RN4)] = RN3 − RN4)
2	Error 2	[y(1,1) − y(1,3)] − [y(2,1)− y(2,4)]	[(LN4 − LN2)] − [(LN3 − LN4)] = (2LN4 − LN2 − LN3)	[(RN2 − RN4)] − [(RN2 −RN1)] = (RN1 − RN4)
3	Error 3	[y(1,1)−y(1,4)]−[y(2,1)−y(2,3)]	[(LN4 − LN3)] − [(LN3 − LN2)] = (LN4+LN2 −2LN3)	[(RN2 − RN1) − [(RN2 −RN3)] = (RN3 −RN1)

Table 7 [RBD-II.2]: Status of Three Error Functions based on Within Block Contrasts of Block 1 and Block 3 - in the presence of Neighbor-Effects for the RBD-II

Sl. No	Error functions	Observational Contrast Based on Block 1 - Block 3	L-N Effects in model expectation	R-N Effects in model expectation
1	Error 4	[y(1,1) − y(1,2)] − [y(3,2) − y(3,1)]	[(LN4 −LN1)] − [(LN2− LN4)] = (2LN4 − LN1 − LN2)	[(RN2−RN3)] − [(RN2−RN4)] = (RN1 + RN2 − 2RN3)
2	Error 5	[y(1,1) − y(1,3)] − [y(3,2) −y(3,3)]	[(LN4−LN2)]−[(LN2−LN1)] = (LN1+LN4 −2LN2)	[(RN2 −RN4)] − [(RN3 - RN4)] = (RN2−RN3)
3	Error 6	[y(1,1) − y(1,4)] − [y(3,2) − y(3,4)]	[(LN4 − LN3)] − [(LN2 – LN3)] = (LN4 −LN2)	[(RN2−RN1)] − [(RN3 − RN2)] = (2RN2 −RN1−RN3)

Table 8 [RBD-II.3]: Status of Three Error Functions based on Within Block Contrasts of Block 1 and Block 4 - in the presence of Neighbor-Effects for the RBD-II

Sl. No	Error functions	Observational Contrast Based on Block 1 - Block 4	L-N Effects in model expectation	R-N Effects in model expectation
1	Error 7	[y(1,1) − y(1,2)] − [y(4,2) − y(4,1)]	[(LN4 − LN1)] − [(LN2 − LN3)] = (LN4 + LN3 − LN1 −LN2)	[(RN2 − RN3)] − [(RN4 – RN1)] = (RN1 + RN2 −RN3−RN4)
2	Error 8	[y(1,1) − y(1,3)] − [y(4,2) − y(4,4)]	[(LN4 − LN2)] − [(LN2 − LN4)] = 2(LN4−LN2)	[(RN2 − RN4)] − [(RN4 − RN2)] = 2(RN2 − RN4)
3	Error 9	[y(1,1) − y(1,4)] − [y(4,2) −y(4,3)]	[(LN4 − LN3)] − [(LN2 − LN1)] = (LN1 + LN4 − LN2−LN3)	[(RN2 − RN1)] − [(RN4 − RN3)] = (RN2 + RN3 − RN1−RN4)

Remark 4: These are rather discouraging results. We fail to estimate the error variance from the so-called $3 \times 3 = 9$ linear error functions based on pair-wise within block contrasts, in the presence of N-effects - both Left-sided and Right-sided. This happened with the RBD-II. We may, however,

derive one positive point from the above results. Upfront, these 9 observational contrasts do provide estimates of contrasts involving both N-Effects - Left-Neighbors and Right-Neighbors. These are useful information towards data analysis. Next, we carry out an analysis of linear observational contrasts based on the four block totals in RBD-II.

Table 9 [RBD-II]: Status of Three Block Total Contrasts in the presence of Neighbor - Effects for the RBD-II

Sl. No	Observational Contrast Based on Block 1 Total - Block j	Total L-N Effects in model expectation	R-N Effects in model expectation
j=2	[y(1,1) + y(1,2) + y(1,3) + y(1,4)] − [y(2,1) + y(2,2) + y(2,3) + y(2,4)]	0	0
j=3	[y(1,1) + y(1,2) + y(1,3) + y(1,4)] − [y(3,1) + y(3,2) + y(3,3) + y(3,4	0	0
j=4	[y(1,1) + y(1,2) + y(1,3) + y(1,4)] − [y(4,1) + y(4,2) + y(4,3) + y(4,4)]	0	0

Remark 5: This is, however, an encouraging result. We are able to estimate block total variances i.e., $\sigma_b^2 + {\sigma_e^2}/{4}$ with 3 df, based on RBD-II data. Finally, we proceed to carry out similar data analysis for RBD-III in respect of error functions.

Table 10 [RBD-III.1]: Status of Three Error Functions based on Within Block Contrasts of Block 1 and Block 2 - in the presence of Neighbor-Effects for the RBD-III

Sl. No	Error functions	Observational Contrast Based on Block 1 - Block 2	L-N Effects in model expectation	R-N Effects in model expectation
1	Error 1	[y(1,1) − y(1,2)] − [y(2,2) − y(2,1)]	[(LN4 − LN1)] − [(LN2-LN3)] = (LN3 + LN4−LN1 − LN2)	[(RN2 − RN3)] − (RN4−RN1)] =(RN1 + RN2 − RN3 − RN4)
2	Error 2	[y(1,1) − y(1,3)] − [y(2,2) − y(2,4)]	[(LN4 − LN2)] − [(LN2 - LN4)] = 2(LN4 - LN2)	[(RN2 −RN4)] − (RN4−RN2)] = 2(RN2−RN4)
3	Error 3	[y(1,1) −y(1,4)] − [y(2,2) − y(2,3)]	[(LN4 − LN3)] − [(LN2 − LN1)] = (LN1 + LN4 −LN2 − LN3)	[(RN2 − RN1)] − [(RN4 RN3)] = (RN2 + RN3 − RN1- RN4)

Table 11 [RBD-III.2]: Status of Three Error Functions based on Within Block Contrasts of Block 1 and Block 3 - in the presence of Neighbor-Effects for the RBD-III

Sl. No	Error functions	Observational Contrast Based on Block 1 - Block 3	L-N Effects in model expectation	R-N Effects in model expectation
1	Error 4	[y(1,1) −y(1,2)] − [y(3,3) −y(3,4)]	0	0
2	Error 5	[y(1,1) − y(1,3)] − [y(3,3) − y(3,1	0	0
3	Error 6	[y(1,1) − y(1,4)] − [y(3,3) − y(3,2)]	0	0

Table 12 [RBD-III.3]: Status of Three Error Functions based on Within Block Contrasts of Block 1 and Block 4 - in the presence of Neighbor-Effects for the RBD-III

Sl. No	Error functions	Observational Contrast Based on Block 1- Block 4	L-N Effects in model expectation	R-N Effects in model expectation
1	Error 7	[y(1,1) − y(1,2)] − [y(4,4) − y(4,3)]	[(LN4 − LN1)] − [(LN2 − LN3)] = (LN3 +LN4 − LN1 − LN2)	[(RN2 − RN3)] −[(RN4 − RN1)] = (RN1 + RN2 − RN3 − RN4)
2	Error 8	[y(1,1) − y(1,3)] − [y(4,4) − (4,2)]	[(LN4 − LN2)] − [(LN2 − LN4)] = 2(LN4 − LN2)	[(RN2 − RN4)] − [(RN4 − RN2)] = 2(RN2 − RN4)
3	Error 9	[y(1,1) − y(1,4)] − [y(4,4) − (4,1)]	[(LN4 − LN3)] − [(LN2 − LN1) = (LN1 + LN4 − LN2 − LN3)	[(RN2 − RN1)] − [(RN4 − RN3)] = (RN2 + RN3 − RN1 −RN4)

Remark 6. These are rather mixed results. We fail to estimate the error variance from all the so-called 3 × 3 = 9 linear error functions based on pair-wise within block contrasts, in the presence of N-effects - both Left-sided and Right-sided. As we see, only the three observational contrasts based on Block 1−Block 3 [Errors 4, 5, 6] do provide estimate of the pure error variance σ_e^2. But there is an interesting feature to be observed too. This is that [Error 1 − Error 7] is a pure error and so are [Error 2 − Error 8] and [Error 3 − Error 9]. To sum it up, based on the six errors, three may be derived for estimation of σ_e^2 while the other three are to be utilized towards estimation of N-Effects contrasts. These facts are to be used in data analysis. Next, we carry out an analysis of linear observational contrasts based on the four block totals in RBD-III.

Table 13 [RBD-III]: Status of Three Block Total Contrasts in the presence of Neighbor - Effects for the RBD-III

Sl. No	Observational Contrast Based on Block 1 Total - Block j Total	L-N Effects in model expectation	R-N Effects in model expectation
j = 2	[y(1,1) + y(1,2) + y(1,3) + y(1,4)] − [y(2,1) + y(2,2) + y(2,3) + y(2,4)]	0	0
j = 3	[y(1,1) + y(1,2) + y(1,3) + y(1,4)] − [y(3,1) + y(3,2) + y(3,3) + y(3,4)]	0	0
j = 4	[y(1,1) + y(1,2) + y(1,3) + y(1,4)] − [y(4,1) + y(4,2) + y(4,3) + y(4,4)]	0	0

Remark 7. It thus transpires that the three block total contrasts are pure errors and these relate to estimation of σ_b^2 + ${\sigma_e^2}/{4}$. Combining the above findings, we can provide estimates of both the variance components. Also we have available estimates of three N-effects contrasts, involving both Left - and Right- Neighbors. These facts are to be used in the combined data analysis.

3. Study of Estimability of Treatment Effects Contrasts

At the last leg, we study the status of observational contrasts - leading to a combination of treatment effects contrasts and possibly also of the N-effects contrasts. From each RBD, these are to be obtained from the considerations of three treatment contrasts viz., $\tau_1 - \tau_2$, $\tau_1 - \tau_3$ and $\tau_1 - \tau_4$.

Not to obscure the essential steps of reasoning, we consider a meaningful representation of the model expectations in terms of $\tau_1 - \tau_2$, $\tau_1 - \tau_3$ and $\tau_1 - \tau_4$, in addition to contrasts involving L-N and R-N Effects. This we do for each RBD.

We refer to Tables 14-15-16 below for the initial analysis regarding estimability of treatment effects contrasts.

We now close our arguments based on the three RBDs one by one.

Table 14 [RBD-I]: Status of Three Treatment Total Contrasts in the presence of Neighbor-Effects for the RBD-I

Sl. No	Model expectation of Observational Contrast Based on Treatment 1 Total − Treatment j Total
j = 2	$4(\tau_1 - \tau_2)$ + 4 (LN4 − LN1 + RN2 − RN3)
j = 3	$4(\tau_1 - \tau_3)$ + 4 (LN4 − LN2) + 4(RN2 − RN4)
j = 4	$4(\tau_1 - \tau_4)$ + 4 (LN4 − LN3) + 4(RN2 − RN1)

It is to be noted that the same expression holds for all the blocks of RBD I, thereby explaining the multiplier 4.

Logical deduction: None of the three treatment effects contrasts is unbiasedly estimable in the presence of the Left and Right N-Effects.

Table 15 [RBD-II]: Status of Three Treatment Total Contrasts in the presence of Neighbor - Effects for the RBD-II

Sl. No	Model expectation of Observational Contrast Based on Treatment 1 Total - Treatment j Total
j = 2	$4(\tau_1 - \tau_2)$ + 2 (LN2 − LN1 + RN2 − RN1)
j = 3	$4(\tau_1 - \tau_3)$ + (LN3 + LN2 − LN1 − LN4) + (RN2 + RN3 − RN1 − RN4)
j = 4	$4(\tau_1 - \tau_4)$ + (LN4 + LN2 − LN1 − LN3) + (RN2 + RN4 − RN1 − RN3)

In order to extract meaningful information about treatment effects contrasts, we show the details below.

Block 1: [(1 2 3 4)] : mean observational contrast leading to
$(\tau_1 - \tau_2)$ + [LN4 − LN1] + [RN2 – RN3]
$(\tau_1 - \tau_3)$ + [LN4 − LN2] + [RN2 – RN4] (1)
$(\tau_1 - \tau_4)$ + [LN4 − LN3] + [RN2 – RN1]

Block 2: [(1 2 4 3)] : mean observational contrast leading to
$(\tau_1 - \tau_2)$ + [LN3 − LN1] + [RN2 – RN4]
$(\tau_1 - \tau_3)$ + [LN3 − LN4] + [RN2 – RN1]

$(\tau_1 - \tau_4)$ + [LN3 $-$ LN2] + [RN2 – RN3] (2)

Block 3: [(2 1 3 4)] : Observational contrast leading to
$(\tau_1 - \tau_2)$ + [LN2 $-$ LN4] + [RN3 – RN1]
$(\tau_1 - \tau_3)$ + [LN2 $-$ LN1] + [RN3 – RN4]
$(\tau_1 - \tau_4)$ + [LN2 $-$ LN3] + [RN3 – RN2] (3)

Block 4 : [(2 1 4 3)] : mean observational contrast leading to
$(\tau_1 - \tau_2)$ + [LN2 $-$ LN3] + [RN4 – RN1]
$(\tau_1 - \tau_3)$ + [LN2 $-$ LN4] + [RN4 – RN2] (4)
$(\tau_1 - \tau_4)$ + [LN2 $-$ LN1] + [RN4 – RN3]

Logical deduction: (1) and (4) together imply estimability of $(\tau_1 - \tau_3)$. Further, (2) and (3) together imply estimability of $(\tau_1 - \tau_4)$. However, estimability of $(\tau_1 - \tau_2)$ is not yet settled in either direction.

One reviewer suggested that we provide a more elaborate argument towards this last statement in either direction and rightly so. There are altogether four observations [one in each block] pertaining to Treatment 1. Let us serially attach coefficients (a, b, c, d) to these observations to form a linear combination. Similarly we attach coefficients (e, f, g, h) to the four observations underlying Treatment 2.

Then for unbiased estimation of the Treatment Contrast $(\tau_1 - \tau_2)$ using the observational contrast based on the two linear combinations suggested above, necessary and sufficient condition are listed below :

(i) a+b+c+d = e+f+g+h =1 [leading to $\tau_1 - \tau_2$]
(ii) a = g, b = h, c = e, d = f, a+b = 0, c+d = 0 [to eliminate Left and Right N-Effects parameters] .

From (i) and (ii), we reach a contradiction.
It is, however, to be noted that we have yet 4+4=8 observations [or observational equations] to be considered in this context. These observations pertain to Treatments 3 and 4 and these may be used to build the so called error functions [in the usual RBD model and in the absence of N-Effects]. We take up the study below.

We develop model expectations of all the 16 observations –four for each treatment. Further, besides those coefficients underlying treatments 1 and 2, we attach coefficients.
(iii) (i, j, k, l) to those underlying Treatment 3 and (iv) (m, n, o, p) to those underlying Treatment 4.

It is a routine task to develop the following list of necessary and sufficient condition for existence of unbiased estimator for the treatment contrast $(\tau_1 - \tau_2)$ based on the 16 observations in the RBD II :
(1) a+b+c+d = $-$ (e+f+g+h) [non-zero];
(2) i+j+k+l = 0, m+n+o+p = 0;
(3) a+g+j+l = 0, b+h+m+o = 0,
(4) c+d+i+n = 0, e+f+k+p = 0;
(5) d+f+i+k = 0, c+e+n+p = 0;
(6) a+b+l+o = 0, g+h+j+m = 0.

One solution is readily found: a=b=c=d=1; e=f=g=h= $-$ 1; ($-$ 1, 1, 1, $-$ 1); (1, $-$ 1, $-$ 1, 1).

Accordingly, mean of the proposed observational contrast is given by 4[$\tau_1 - \tau_2$]. Hence, ($\tau_1 - \tau_2$) is indeed estimable as well. Thus for RBD II, all the three treatment contrasts are estimable, even in the presence of N-Effects on both sides.

Table 16 [RBD-III]: Status of Three Treatment Total Contrasts in the presence of Neighbor - Effects for the RBD-III

Sl. No	Model expectation of Observational Contrast Based on Treatment 1 Total − Treatment j Total
j = 2	4($\tau_1 - \tau_2$) + 2(LN4 + LN2 − LN1 − LN3) + 2(RN2 + RN4 − RN1 − RN3)
j = 3	4($\tau_1 - \tau_3$)
j = 4	4($\tau_1 - \tau_4$) + 2(LN4 + LN2 − LN1 − LN3) + 2(RN2 + RN4 − RN1 − RN3)

We now show further details for settling the issues of estimability of treatment contrasts in RBD - III.

Block 1: [(1 2 3 4)]: mean model yields

Tr.1	1 2 3 4
LN	4 1 2 3
RN	2 3 4 1

Block 2: [(2 1 4 3): mean model yields

Tr.1	1 2 3 4
LN	2 3 4 1
RN	4 1 2 3

Block 3: [(3 4 1 2): mean model yields

Tr.1	1 2 3 4
LN	4 1 2 3
RN	2 3 4 1

Block 4: [(4 3 2 1): mean model yields

Tr.1	1 2 3 4
LN	2 3 4 1
RN	4 1 2 3

Estimation of treatment contrasts:

It readily follows from the above that

(a) an observational contrast based on means underlying Treatment 1 and Treatment 3 is free from all the N-Effects in both directions. Hence $\tau_1 - \tau_3$ is estimable;

(b) an observational contrast based on means underlying Treatment 2 and Treatment 4 is also free from all the N-Effects in both directions. Hence $\tau_2 - \tau_4$ is also estimable.

However, $\tau_1 - \tau_2$ is not estimable based on contrasts involving the treatment means. We may use close critical argument to establish this. Once more, we use the coefficients [(a, b, c, d); (e, f, g, h); (i, j, k, l); (m, n, o, p)] as before. Then we need the following conditions to be satisfied:

(i) a+b+c+d+e+f+g+h = 0; (ii) i+j+k+l = 0; (iii) m+n+o+p = 0; (iv) a+c+j+l = 0; (vi) b+d+i+k = 0; (vii) e+g+n+p = 0; (viii) f+h+m+o = 0.

These together imply: a+b+c+d = 0 = e+f+g+h.

Hence we arrive at a contradiction.

Remark 8: It thus transpires that from the treatment effects contrasts of RBD-I, we are unable to provide any unbiased estimates of any of them - independent of N-effects contrasts. From RBD-II, it follows that all the treatment contrasts are estimable independent of N-effects. Besides, in RBD-

III, $\tau_1 - \tau_3$ and $\tau_2 - \tau_4$ are both estimable however $(\tau_1 - \tau_2)$ is not estimable. These facts are to be used in the combined data analysis.

4. Study of Estimability of Left- and Right-Neighbor Effects Contrasts

We set α_1 = LN1 − LN2; α_2 = LN1 − LN3; α_3 = LN1 − LN4 and δ_1= RN1 − RN2; δ_2 = RN1 − RN3; δ_3 = RN1 − RN4. Then we can express LN- and RN- Effects contrasts in terms of the αs and δs. In the above detailed analysis of the RBDs, we have identified observational contrasts that involve these parameters. We first list them below.

Table 17 [RBD-I]: Sources of N-Effects in Observational Contrasts

Sl. No	Observational Contrasts Based on Treatment Contrasts	L-N Effects in model expectation	R-N Effects in model expectation
1	$\tau_1 - \tau_2$	$-\alpha_3$	$-\delta_1 + \delta_2$
2	$\tau_1 - \tau_3$	$\alpha_1 - \alpha_3$	$-\delta_1 + \delta_3$
3	$\tau_1 - \tau_4$	$\alpha_2 - \alpha_4$	$-\delta_1$

Table 18 [RBD-II]: Sources of N-Effects in Observational Contrasts

Sl. No	Observational Contrasts Based on Treatment Contrasts	L-N Effects in model expectation	R-N Effects in model expectation
1	$\tau_1 - \tau_2$	$-2\alpha_1$	$-2\delta_1$
2	$\tau_1 - \tau_3$	$-\alpha_1 - \alpha_2 + \alpha_3$	$-\delta_1 - \delta_2 + \delta_3$
3	$\tau_1 - \tau_4$	$-\alpha_1 + \alpha_2 - \alpha_3$	$-\delta_1 + \delta_2 - \delta_3$

Table 19 [RBD-III]: Sources of N-Effects in Observational Contrasts

Sl. No	Observational Contrasts Based on Treatment Contrasts	L-N Effects in model expectation	R-N Effects in model expectation
1	$\tau_1 - \tau_2$	$-2\alpha_1 + 2\alpha_2 - 2\alpha_3$	$-2\delta_1 + 2\delta_2 - 2\delta_3$
2	$\tau_1 - \tau_3$	0	0
3	$\tau_1 - \tau_4$	$-2\alpha_1 + 2\alpha_2 - 2\alpha_3$	$-2\delta_1 + 2\delta_2 - 2\delta_3$

Table 20 [RBD-II]: Additional Sources of N-Effects in Observational Contrasts

Sl. No	Observational Contrasts Based on errors	L-N Effects in model expectation	R-N Effects in model expectation
1	Error 1	$\alpha_2 - \alpha_3$	$-\delta_2 + \delta_3$
2	Error 2	$\alpha_1 + \alpha_2 - 2\alpha_3$	δ_3
3	Error 3	$-\alpha_1 + 2\alpha_2 - \alpha_3$	$-\delta_2$
4	Error 4	$\alpha_1 - 2\alpha_3$	$-\delta_1 + 2\delta_2$
5	Error 5	$2\alpha_1 - \alpha_3$	$-\delta_1 + \delta_2$
6	Error 6	$\alpha_1 - \alpha_3$	$-2\delta_1 + \delta_2$
7	Error 7	$\alpha_1 - \alpha_2 - \alpha_3$	$-\delta_1 + \delta_2 + \delta_3$
8	Error 8	$2\alpha_1 - 2\alpha_3$	$-2\delta_1 + 2\delta_3$
9	Error 9	$\alpha_1 + \alpha_2 - \alpha_3$	$-\delta_1 - \delta_2 + \delta_3$

Table 21 [RBD-III]: Additional Sources of N-Effects in Observational Contrasts

Sl. No	Observational Contrasts Based on errors	L-N Effects in model expectation	R-N Effects in model expectation
1	Error 1	$\alpha_1 - \alpha_2 - \alpha_3$	$-\delta_1 + \delta_2 + \delta_3$
2	Error 2	$2\alpha_1 - 2\alpha_3$	$-2\delta_1 + 2\delta_3$
3	Error 3	$\alpha_1 + \alpha_2 - \alpha_3$	$-\delta_1 + \delta_2 + \delta_3$
4	Error 7	$\alpha_1 - \alpha_2 - \alpha_3$	$-\delta_1 + \delta_2 + \delta_3$
5	Error 8	$2\alpha_1 - 2\alpha_3$	$-2\delta_1 + 2\delta_3$
6	Error 9	$\alpha_1 + \alpha_2 - \alpha_3$	$-\delta_1 - \delta_2 + \delta_3$

Remark 9: With reference to RBD-II in Table 18, it may be noted that $\tau_1 + \tau_2 - \tau_3 - \tau_4$ is free from the N-Effects and so it is unbiasedly estimable. Therefore, we should consider any one between the first and the third in the above list for analysis wrt estimation of the N-Effects cntrasts.
Remark 10: With reference to RBD-III in Table 19, it may again be easily noted that $\tau_1 - \tau_3$ and $\tau_2 - \tau_4$ are both estimable. Therefore, we should consider any one of the non-null two cases in the above list for analysis wrt estimation of the N-Effects contrasts.
Remark 11: It must be noted that there are some additional sources of observational contrasts leading to N-Effects contrasts. We now list them below in Tables 20 and 21.
Remark 12: It follows readily from the list of six observational contrasts [based on RBD-III] in Table 21 that we may select, say the upper half of three contrasts for the purpose of building up a model for estimation of LN- and RN-Effects contrasts.
On final count, we list below all the 17 observational contrasts for each of which the model expectation involves exclusively αs and δs.

X =

0	0	-1	-1	1	0
1	0	-1	-1	0	1
0	1	-1	-1	0	0
1	0	0	1	0	0
-1	1	-1	-1	1	-1
0	1	-1	0	-1	1
1	1	-2	0	0	1
-1	2	-1	0	-1	0
1	0	-2	-1	2	0
2	0	-1	-1	1	0
1	0	-1	-2	1	0
1	-1	-1	-1	1	1
1	0	-1	-1	0	1
1	1	-1	-1	-1	1
1	-1	-1	-1	1	1
1	0	-1	-1	0	1
1	1	-1	-1	-1	1

5. Concluding Remark In this paper, we have addressed and resolved interesting issues in the context of data analysis underlying RBDs in a mixed-effects model when there are Left- and

Right-Neighbor effects. We have clearly demonstrated that the data analysis is non-trivial in such cases. The choice of three RBDs and their joint study pose interesting data analysis problems. Study of existence and visualization of error functions in such contexts is a highly non-trivial exercise. Our expertise in this area of research is amply evident from our published work cited in the references. We exactly know the gaps in this area of research and we have tried to fill in the gaps – even by taking a specific case of RBD with b = v = 4. This may qualify as a 'case study'. However, this itself poses interesting issues as we go along. This study also reveals that our usual understanding about linear error functions doesn't work in such models and in the presence of Left- and Right-Neighbor effects.

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