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An Integral Proof of the Joint M.G.F. of Sample Mean and Variance

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Abstract

Without assuming independence of sample mean and variance, or without using any conditional distribution, we present an integral proof of the joint moment generating function for sample mean and variance for independently, identically and normally distributed random variables. This proves the independence of sample mean and variance which is the basis of Student t-test and many other inferential methods.

Keywords: Sample mean, sample variance, independence, moment generating function

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1. Introduction

The independence of sample mean and variance of independently, identically and normally distributed variables is essential in the basic definition of Student t-statistic, and also in the development of many statistical methods. It is usually proved using the independence of $\bar{X}$ and $(X_1-\bar{X},\ X_2-\bar{X},\ \ldots, X_n-\bar{X})$, (see e.g. Theorem 1, p.340, Rohatgi and Saleh, 2001), but this requires background on independence of functions of random variables (Theorem 2, p.121, Rohatgi and Saleh, 2001). Shuster (1973) and Zehna (1991) use moment generating function to prove the independence of sample mean and variance. In this note we give a new proof of this independence that also uses moment generating function, but it avoids the use of conditional distributions though it requires background in multivariable integral calculus.

2. Some Preliminaries

Let $X_1, X_2, \ldots, X_n$ $(n=2,3,\cdots)$ have an arbitrary n-dimensional joint distribution. We define the sample mean $\bar{x}$ and variance s^2 by $n\bar{x}=\sum_{i=1}^{i=n} x_i$ and $(n-1)s^2=\sum_{i=1}^{i=n}(x_i-\bar{x})^2$, respectively. The sample variance can also be represented by

$$n(n-1)s^2 = (n-1)\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n} x_i x_j. \tag{2.1}$$

Also for identically distributed observations $X_1, X_2, \ldots, X_n$ with common mean $\mu,$ we denote $\mu'_a \equiv E(X^a),$ the a-th moment of X and $\mu_a \equiv E(X-\mu)^a$, the centered moment of X order a. The mean μ'_1 and variance $\mu_2 \equiv V(X)$ will be simply denoted by μ and $\sigma^2 = \mu'_2 - \mu^2$ respectively.

The moment generating function of $X \sim N(\mu, \sigma^2)$ is given by

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right), \quad -\infty < t < \infty.$$

Let $A = [a_{ij}]$ be a $n \times n$ positive definite symmetric matrix. Then the following integral is known:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} a_{ij}x_i x_j + \sum_{i=1}^{i=n} b_i x_i\right) dx_1 dx_2 \cdots dx_n = \frac{(2\pi)^{n/2}}{|A|^{1/2}} \exp\left(\frac{1}{2}\underset{\sim}{b}' A^{-1} \underset{\sim}{b}\right), \tag{2.2}$$

where $\underset{\sim}{b} = (b_1, b_2, \cdots b_n)'$. The above expression usually appears as part of the moment generating function

$$M_{\underset{\sim}{X}}(\underset{\sim}{b}) = \exp\left(\frac{1}{2}\underset{\sim}{b}'\Sigma\underset{\sim}{b}\right), \tag{2.3}$$

of a random variable $\underset{\sim}{X}$ having multivariate normal distribution $N(\underset{\sim}{0}, \Sigma)$. See for example, Anderson (1984, p.21 and p.47).

For the proof of the following theorem dealing with the properties of a pattern matrix, see Rao (1973, p.67).

Theorem 2.1 Let the $n \times n$ matrix $\Gamma = [\gamma_{ij}]$ where $\gamma_{ii} = \alpha$, for $i = 1, 2, \cdots, n,$ and $\gamma_{ij} = \beta$, for $i = 1, 2, \cdots, n,$ and $j(\neq i) = 1, 2, \cdots, n.$ Then the following hold:

a. $|\Gamma| = (\alpha - \beta)^{n-1}[\alpha + (n-1)\beta],$

b. Γ^{-1} exists if and only if $\alpha \neq \beta$ and $\alpha \neq (1-n)\beta$. Moreover, if γ^{ii} and γ^{ij} are the entries of Γ^{-1}, then

$$\gamma^{ii} = \frac{\alpha + (n-2)\beta}{[\alpha + (n-1)\beta](\alpha - \beta)}, \text{ and } \gamma^{ij} = \frac{-\beta}{[\alpha + (n-1)\beta](\alpha - \beta)}, \; i \neq j.$$

Corollary 2.1 Let the $n \times n$ matrix $\Gamma = [\gamma_{ij}]$ where $\gamma_{ii} = \alpha$, for $i = 1, 2, \cdots, n,$ and $\gamma_{ij} = \beta,$ for $i = 1, 2, \cdots, n,$ and $j (\neq i) = 1, 2, \cdots, n.$ Further if $\alpha + (n-1)\beta = 1,$ then we have the following:

a. $|\Gamma| = (\alpha - \beta)^{n-1},$

b. $\sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \gamma_{ij} = n,$

c. $\sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \gamma^{ij} = n.$

Proof.

a. It is obvious from part (a) of Theorem 2.1.

b. Since $\alpha + (n-1)\beta = 1,$ it follows that

$\sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} = \sum_{i=1}^{n} \gamma_{ii} + \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} \gamma_{ij}$ which equals $n.$

c. Since $\alpha + (n-1)\beta = 1,$ it follows from Theorem 2.1 that $\gamma^{ii} = \dfrac{1-\beta}{\alpha - \beta}, \quad (i = 1, 2, \cdots, n)$

and $\gamma^{ij} = \dfrac{-\beta}{\alpha - \beta}, \ (i = 1, 2, \cdots, n, \ j(\neq i) = 1, 2, \cdots, n).$ Then

$\sum_{i=1}^{n} \sum_{j=1}^{n} \gamma^{ij} = \sum_{i=1}^{n} \gamma^{ii} + \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} \gamma^{ij}$ which equals $n.$

3. The Joint M.G.F. of Sample Mean and Variance

Without using any conditional distribution or assuming independence of $\bar{X}$ and $S^2,$ we present a direct proof of the joint moment generating function of $\bar{X}$ and S^2, based on independently, identically and normally distributed random variables.

Theorem 3.1 Let the random variables $X_1, X_2, \cdots, X_n, \ (n \geq 2)$ be independently, identically and normally distributed with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2, \ \sigma > 0.$ Then the joint moment generating function of the sample mean $\bar{X}$ and variance S^2 is given by

$$M_{\bar{X}, S^2}(t_1, t_2) = \left[\exp\left(\mu t_1 + \frac{\sigma^2 t_1^2}{2n}\right)\right]\left(1 - \frac{2\sigma^2 t_2}{n-1}\right)^{-(n-1)/2}, \quad n \geq 2. \tag{3.1}$$

In particular, $\bar{X}$ and S^2 are independent, and $(n-1)S^2 / \sigma^2 \sim \chi^2_{n-1}.$

Proof. The joint moment generating function of $\bar{X}$ and S^2 is given by

$$M_{\bar{X},S^2}(t_1,t_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(t_1\bar{x}+t_2s^2\right)\prod_{i=1}^{n}f(x_i)dx_i,$$

which equals

$$M_{\bar{X},S^2}(t_1,t_2)=\frac{1}{(2\pi)^{n/2}\sigma^n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(t_1\bar{x})\exp\left(t_2s^2\right)\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right]\prod_{i=1}^{n}dx_i. \quad (3.2)$$

From (2.1), we have $s^2=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}x_ix_j$ so that

$$\exp\left(t_2s^2\right)=\exp\left(\frac{t_2}{n}\sum_{i=1}^{n}x_i^2-\frac{t_2}{n(n-1)}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}x_ix_j\right).$$

Then the expression (3.2) can be written as

$$\begin{aligned}M_{\bar{X},S^2}(t_1,t_2)=&\frac{1}{(2\pi)^{n/2}\sigma^n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(t_1\bar{x})\exp\left(\frac{t_2}{n}\sum_{i=1}^{n}x_i^2-\frac{t_2}{n(n-1)}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}x_ix_j\right)\\&\times\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right]\prod_{i=1}^{n}dx_i.\end{aligned} \quad (3.3)$$

Using the representation of s^2 given in (2.1) in the above integral, and the transformation $x_i=\mu+\sigma z_i,\ (i=1,2,\cdots,n)$ yields

$$M_{\bar{X},S^2}(t_1,t_2)=\frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp[t_1(\mu+\sigma\bar{z})]\exp\left(\frac{\sigma^2t_2}{n}\sum_{i=1}^{n}z_i^2-\frac{\sigma^2t_2}{n(n-1)}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}z_iz_j\right)\exp\left(-\frac{1}{2}\sum_{i=1}^{n}z_i^2\right)\prod_{i=1}^{n}dz_i.$$

The joint M.G.F. of the sample mean and variance can then be written as

$$M_{\bar{X},S^2}(t_1,t_2)=\frac{\exp(t_1\mu)1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(t_1\sigma\bar{z})\exp\left(-\frac{1}{2}\sum_{i=1}^{n}z_i^2+\frac{\sigma^2t_2}{n}\sum_{i=1}^{n}z_i^2-\frac{\sigma^2t_2}{n(n-1)}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}z_iz_j\right)\prod_{i=1}^{n}dz_i$$

or,

$$M_{\bar{X},S^2}(t_1,t_2)=\frac{\exp(t_1\mu)1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(\sum_{i=1}^{n}b_iz_i\right)\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\gamma_{ii}z_i^2-\frac{1}{2}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\gamma_{ij}z_iz_j\right)\prod_{i=1}^{n}dz_i,$$

where $\underset{\sim}{b}$ is the n-vector $\frac{\sigma t_1}{n}\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}'$, and $\Gamma=(\gamma_{ij})$ be the $n\times n$ matrix given by

$$\gamma_{ij}=\begin{cases}1-\dfrac{2\sigma^2t_2}{n}, & \text{if } i=j\\ \dfrac{2\sigma^2t_2}{n(n-1)}, & \text{if } i\neq j.\end{cases}$$

The above can also be written as the following:

$$M_{\bar{X},S^2}(t_1,t_2)=\frac{\exp(t_1\mu)}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(\sum_{i=1}^{n}b_i z_i\right)\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}z_i z_j\right)\prod_{i=1}^{n}dz_i. \tag{3.4}$$

The expression (3.4) can then be written as

$$M_{\bar{X},S^2}(t_1,t_2)=\frac{\exp(t_1\mu)}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\underset{\sim}{z}'\Gamma\underset{\sim}{z}\right)\exp(\underset{\sim}{b}'\underline{z})d\underset{\sim}{z},$$

which, by (2.3), can be evaluated to be

$$M_{\bar{X},S^2}(t_1,t_2)=\exp(t_1\mu)\,|\,\Gamma\,|^{-1/2}\exp\left(\frac{1}{2}\underset{\sim}{b}'\Gamma^{-1}\underset{\sim}{b}\right). \tag{3.5}$$

It is easy to check that

$$\underset{\sim}{b}'\Gamma^{-1}\underset{\sim}{b}=tr\Gamma^{-1}\underset{\sim}{b}\underset{\sim}{b}'=\frac{\sigma^2 t_1^2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma^{ij}=\frac{\sigma^2 t_1^2}{n}, \tag{3.6}$$

where the last step follows by Corollary 2.1(c). Also by Corollary 2.1(a), we have

$|\,\Gamma\,|^{-1/2}=(\alpha-\beta)^{(n-1)/2}$ where $\alpha=1-\dfrac{2\sigma^2 t_2}{n}$ and $\beta=\dfrac{2\sigma^2 t_2}{n(n-1)}$. Since $\alpha-\beta=1-\dfrac{2\sigma^2 t_2}{n-1}$,

we obtain

$$|\,\Gamma\,|^{-1/2}=\left(1-\frac{2\sigma^2 t_2}{n-1}\right)^{-(n-1)/2}. \tag{3.7}$$

Plugging (3.6) and (3.7) in (3.5), we obtain (3.1).

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