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# Bayes Estimation of a Common Mean of Several Normal Populations with Unknown Variances

**Peter M. Mphekgwana[1,2], Yehenew G. Kifle[3*] and Chioneso S. Marange[1]**

[1]Department of Statistics, Faculty of Science and Agriculture,
Fort Hare University, South Africa
[2]Department of Research Administration and Development,
University of Limpopo, South Africa
[3]Department of Mathematics and Statistics,
University of Maryland Baltimore County, USA

Correspondence should be addressed to Yehenew G. Kifle
(Email: yehenew@umbc.edu)

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## Abstract

Combining information from several independent normal populations to estimate a common mean parameter has applications in meta-analysis and is an important statistical problem. For this application, Gregurich and Broemeling (1997) and Tu (2012) concentrated on point estimation employing Bayesian techniques to infer about the common mean of two normal populations with unknown variances. In our study, we expand upon their investigation to encompass $k$ normal populations with a common mean, incorporating a range of objective priors. Through the use of two examples, it is discovered that as the hyperparameter $\alpha$ under a Bayesian framework increases, the performance of the Bayesian technique also improves.

**Keywords:** Meta-analysis; Bayes estimation; Common mean; Noninformative prior.

**AMS Classification:** 62F15.

## 1. Introduction

As medical knowledge continues to grow quickly, healthcare providers are now confronted with substantial difficulties in comprehensively evaluating and analyzing the

relevant data required for making well-informed decisions [1, 2]. Moreover, the diverse and occasionally conflicting results shown in various studies exacerbate this challenge. To mitigate these issues, Gene V. Glass (1976) coined the term "meta-analysis" to describe the statistical examination of a comprehensive assembly of findings from individual studies, aimed at consolidating those results. This approach accomplishes this goal by using sound statistical procedures to a large collection of analysis results from individual studies to combine the findings [3, 2]. Moreover, meta-analysis has garnered considerable interest across various scientific domains, including education and medicine. For instance, in the educational realm, it has been employed to amalgamate studies on the effectiveness of coaching in enhancing Scholastic Aptitude Test (SAT) scores in both verbal and mathematical domains [4]. Similarly, in social sciences, it has been utilized to combine research on gender disparities across quantitative, verbal, and visual-spatial abilities [5]. Furthermore, in healthcare, meta-analysis has proven invaluable, particularly in the context of the COVID-19 pandemic, by enhancing our understanding of the virus's implications and informing public health strategies on the basis of combining evidence from many studies [6, 7].

The problem of combining two or more unbiased estimators arises frequently in applied statistics, where it has important implications in a wide range of fields. A well-known context of this problem occurred when Meier [8] was asked to draw inferences about the mean of albumin in plasma protein in human subjects based on results from four experiments. Another scenario happened when Eberhardt et al. [9] had results from four experiments about non-fat milk powder and the problem was to draw inferences about the mean Selenium in non-fat milk powder by combining the results from four methods.

The early literature predominantly focuses on inferring the common mean $\mu$, primarily concerning point estimation and theoretical decision-making criteria regarding $\mu$. Graybill and Deal (1959) laid the groundwork for this research and proposed what is now known as the Graybill-Deal estimate, which has since seen extensive development and elaboration through subsequent studies [10], as well as additional related references [11, 12]. Furthermore, more work has been done by Meier [8], Sinha [13], and Hartung [14] which significantly contributed to the field by devising methods to approximate confidence intervals for the unbiased estimator of the variance of Graybill-Deal estimate. Additionally, Cohen and Sackrowitz [15], Fairweather [16], Jordan and Krishnamoorthy [17], and Yu, Sun, and Sinha [18] have proposed methods of constructing an exact $100(1-\alpha)\%$ confidence interval for $\mu$ by utilizing appropriate linear combinations of test statistics or $P$-value based functions serving as pivotal quantities for this purpose.

Although meta-analysis is widely recognized in the medical field, its application is not limited to this domain. In fields such as economics and social sciences, Bayesian meta-analysis is gaining popularity due to its ability to incorporate prior knowledge and address heterogeneity in a more nuanced way compared to classical methods. In economics, Bayesian meta-analysis has been effectively employed to synthesize studies

on the impacts of economic policies, such as the effects of minimum wage increases on employment [19]. By incorporating prior information and accounting for variations across different economic contexts, Bayesian methods offer a flexible approach that can provide more refined policy insights compared to classical meta-analytic techniques. In social sciences, Bayesian meta-analysis is particularly advantageous for synthesizing studies with significant heterogeneity, such as those examining the effectiveness of educational interventions[20]. By modeling the uncertainty and variations across different study populations and designs, Bayesian approaches offer more tailored and credible estimates, enhancing the ability to generalize findings across diverse contexts.

The study by Gregurich and Broemeling (1997) and Tu (2012) focused on point estimation and hypothesis testing to draw inferences on the common mean of two normal populations [21, 22]. In this paper, we extend their work to encompass $k$ diverse normal populations, employing varied objective priors.

## 2. Methods

Estimating the common mean of several populations with unknown variance poses a significant challenge in statistical analysis. However, Bayesian approaches offer a flexible alternative that can effectively handle uncertainty in parameter estimation by incorporating prior knowledge. Gregurich and Broemeling (1997) and Tu (2012) suggest that a Bayesian approach holds promise for estimating common mean among several normal populations with unknown variances [21, 22]. Their recommendation stems from the understanding that the outcomes of a meta-analysis, where data from multiple studies are combined to draw overarching conclusions, heavily hinge on the prior information available. In this context, Bayesian techniques offer a distinct advantage by allowing the integration of prior knowledge into the analysis process.

The study conducted by the above authors concentrated on point estimation employing Bayesian techniques to infer the common mean of two normal populations with unknown variances. In our study, we expand upon their investigation to encompass $k$ several normal populations, incorporating a range of subjective priors. To formulate the present problem, we assume only that there are $k$ normal populations with a common mean but with unknown possibly unequal variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2 > 0$. Let us assume that we have independent and identically distributions (i.i.d) observations $X_{i1}, ..., X_{in_i}$ from $N(\mu, \sigma_i^2)$, $i = 1, 2, \ldots, k$ the likelihood function is

$$\prod_{ij} f(x_{ij}|\mu, \sigma_i^2) \propto \left( \prod_i \sigma_i^{-n_i} \right) \exp \left\{ -\sum_{i=1}^{k} \left( \frac{1}{2\sigma_i^2} \right) \sum_{j=1}^{n_i} (x_{ij} - \mu)^2 \right\}. \tag{1}$$

The log-likelihood function of the likelihood given in (1) can be defined as

$$\ell = -\frac{n_1}{2}(ln)\sigma_1^2 - \frac{1}{2\sigma_1^2}\left(\sum_{j=1}^{n_1}(x_j-\mu)^2\right) - \frac{n_2}{2}(ln)\sigma_2^2 - \frac{1}{2\sigma_2^2}\left(\sum_{j=1}^{n_2}(x_j-\mu)^2\right) - \ldots$$
$$- \frac{n_k}{2}(ln)\sigma_k^2 - \frac{1}{2\sigma_k^2}\left(\sum_{j=1}^{n_k}(x_j-\mu)^2\right). \quad (2)$$

Therefore the second derivative of $\ell$ concerning each possible pair of parameters is

$$\begin{aligned}
\frac{\partial^2\ell}{\partial\mu^2} &= -\frac{n_1}{\sigma_1^2} - \frac{n_2}{\sigma_2^2} - \cdots - \frac{n_k}{\sigma_k^2} \\
\frac{\partial^2\ell}{\partial\sigma_1^2} &= \frac{n_1}{2}\left(\frac{1}{\sigma_1^2}\right)^2 - \left(\frac{1}{\sigma_1^2}\right)^3\left(\sum_{j=1}^{n_1}(x_j-\mu)^2\right) \\
\frac{\partial^2\ell}{\partial\sigma_2^2} &= \frac{n_2}{2}\left(\frac{1}{\sigma_2^2}\right)^2 - \left(\frac{1}{\sigma_2^2}\right)^3\left(\sum_{j=1}^{n_2}(x_j-\mu)^2\right) \\
&\vdots \\
\frac{\partial^2\ell}{\partial\sigma_k^2} &= \frac{n_k}{2}\left(\frac{1}{\sigma_k^2}\right)^2 - \left(\frac{1}{\sigma_k^2}\right)^3\left(\sum_{j=1}^{n_k}(x_j-\mu)^2\right).
\end{aligned}$$

Fisher Information matrix of $(\mu, \sigma_1, \sigma_2, \ldots, \sigma_k)$ is

$$I(\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2) = \begin{pmatrix} \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \cdots + \frac{1}{\sigma_k^2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2\sigma_1^4} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2\sigma_2^4} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\sigma_k^4} \end{pmatrix}.$$

Since $I(\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2)$ has a diagonal form, we can calculate Jeffrey's [23] independence prior as

$$\pi(\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2) = \pi(\mu)\pi(\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2) \quad (3)$$

Assuming constant prior for the common mean $\mu$, and the unknown and possibly unequal variances $\sigma_i^2, i = 1, 2, \cdots, k$, two non-informative priors are chosen. It is easy to see the references prior

$$\pi(\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2) \propto \prod_{i=1}^{k} \frac{1}{\sigma_i^{\alpha}}, \quad (4)$$

these prior assume $\mu$ and $\sigma_1, \sigma_2, \ldots, \sigma_k$ are independent and the prior for common mean $\mu$ as $\pi(\mu) = 1$ and for variances $\sigma_i^2, i = 1, 2, \cdots, k$ as $\pi(\sigma_i^2) = \sigma_i^{-\alpha}$. Combining this prior with the likelihood function yields the posterior probability density function (pdf) for $\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$

$$\begin{aligned}
\pi\ (\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2|\mathbf{x}\ ) &= \frac{\pi\ (\mathbf{x}|\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2\ )\pi(\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2)}{\pi\ (\ \mathbf{x}\ )} \\
&\propto \pi\ (\mathbf{x}|\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2\ )\pi(\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2) \\
&\propto \left(\prod_{ij}^{k} \sigma_i^{-n_i}\right) \exp\left\{-\sum_{i=1}^{k}\left(\frac{1}{2\sigma_i^2}\right)\sum_{j=1}^{n_i}\ (x_{ij} - \mu\ )^2\right\}\prod_{i=1}^{k}\frac{1}{\sigma_i^{\alpha}} \\
&\propto \left(\frac{1}{\sigma_1^2}\right)^{\frac{n_1+\alpha}{2}}\left(\frac{1}{\sigma_2^2}\right)^{\frac{n_2+\alpha}{2}}\cdots\left(\frac{1}{\sigma_k^2}\right)^{\frac{n_k+\alpha}{2}} \exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{(x_{ij} - \mu\ )^2}{2\sigma_i^2}\right\}
\end{aligned}$$

From the joint posterior distribution $\pi\ (\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2|\mathbf{x}\ )$, it is possible to derive the conditional posterior distribution, of $\mu$ given the variance components $\sigma_i^2$

$$\begin{aligned}
\pi\ (\mu|\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2, \mathbf{x}\ ) &= \frac{\pi\ (\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2|\mathbf{x}\ )}{\pi\ (\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2\ )} \\
&\propto \left(\sum_{i=1}^{k}\frac{n_i}{\sigma_i^2}\right)^{-1} \exp\left\{-\frac{\left(\sum_{i=1}^{k}\frac{n_i}{\sigma_i^2}\right)^{-1}}{2}\left(\mu - \frac{\sum_{i=1}^{k}\frac{n_i}{\sigma_i^2}\overline{x}_i}{\sum_{i=1}^{k}\frac{n_i}{\sigma_i^2}}\right)\right\}.
\end{aligned}$$

$$\begin{aligned}
\pi\ (\sigma_1^2|\mu, \mathbf{x}\ ) &= \frac{\pi\ (\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2|\mathbf{x}\ )}{\pi\ (\mu, \sigma_2^2, \ldots, \sigma_k)} \\
&= \left[\Gamma\left(\frac{n_1 + \alpha - 2}{2}\right)\right]^{-1}\left(\sum_{j=1}^{n_1}\frac{(x_j - \mu\ )^2}{2}\right)^{\frac{n_1+\alpha-2}{2}}\left(\frac{1}{\sigma_1^2}\right)^{\frac{n_1+\alpha}{2}} \exp\left\{-\frac{1}{\sigma_1^2}\sum_{j=1}^{n_1}\frac{(x_j - \mu\ )^2}{2}\right\} \\
\pi\ (\sigma_2^2|\mu, \mathbf{x}\ ) &= \frac{\pi\ (\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2|\mathbf{x}\ )}{\pi\ (\mu, \sigma_1^2, \ldots, \sigma_k^2)} \\
&= \left[\Gamma\left(\frac{n_2 + \alpha - 2}{2}\right)\right]^{-1}\left(\sum_{j=1}^{n_2}\frac{(x_j - \mu\ )^2}{2}\right)^{\frac{n_2+\alpha-2}{2}}\left(\frac{1}{\sigma_2^2}\right)^{\frac{n_2+\alpha}{2}} \exp\left\{-\frac{1}{\sigma_2^2}\sum_{j=1}^{n_2}\frac{(x_j - \mu\ )^2}{2}\right\} \\
&\vdots \\
\pi\ (\sigma_k^2|\mu, \mathbf{x}\ ) &= \frac{\pi\ (\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2|\mathbf{x}\ )}{\pi\ (\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_{k-1}^2)} \\
&= \left[\Gamma\left(\frac{n_k + \alpha - 2}{2}\right)\right]^{-1}\left(\sum_{j=1}^{n_k}\frac{(x_j - \mu\ )^2}{2}\right)^{\frac{n_k+\alpha-2}{2}}\left(\frac{1}{\sigma_2^2}\right)^{\frac{n_k+\alpha}{2}} \exp\left\{-\frac{1}{\sigma_k^2}\sum_{j=1}^{n_k}\frac{(x_j - \mu\ )^2}{2}\right\}
\end{aligned}$$

The conditional posterior distribution of $\mu$ given the variance $\sigma_i^2$ components still

follows a normal distribution

$$\mu|\sigma_1^2,\sigma_2^2,\ldots,\sigma_k^2,\mathbf{x}\sim N\left\{\frac{\sum_{i=1}^k \frac{n_i}{\sigma_i^2}\overline{x}_i}{\sum_{i=1}^k \frac{n_i}{\sigma_i^2}},\left(\sum_{i=1}^k \frac{n_i}{\sigma_i^2}\right)^{-1}\right\}. \tag{5}$$

The conditional posterior distribution of $\sigma_i^2$ given the $\mu$ variance components follows an Inverse Gamma distribution

$$\sigma_1^2|\mu,\mathbf{x}\sim \text{IG}\left(\frac{n_1+\alpha-2}{2},\sum_j^{n_1}\frac{(x_j-\mu\ )^2}{2}\right). \tag{6}$$

$$\sigma_2^2|\mu,\mathbf{x}\sim \text{IG}\left(\frac{n_2+\alpha-2}{2},\sum_j^{n_2}\frac{(x_j-\mu\ )^2}{2}\right). \tag{7}$$

$$\vdots$$

$$\sigma_k^2|\mu,\mathbf{x}\sim \text{IG}\left(\frac{n_k+\alpha-2}{2},\sum_j^{n_k}\frac{(x_j-\mu\ )^2}{2}\right). \tag{8}$$

Since $\sum_i\ (x_i-\mu\ )^2=(n_i-1)s_i^2+n_i(\overline{x}_i-\mu)^2$, it is clear that we do not need micro data, but only the summary statistics $\overline{x}$ and $s^2$.

**Remark 1**. For the case when $\alpha=2$, prior for $\mu,\sigma_1^2,\sigma_2^2,\ldots,\sigma_k^2$ is $\pi(\mu,\sigma_1^2,\sigma_2^2,\ldots,\sigma_k^2)=\prod_{i=1}^k\sigma_i^{-2}$. The conditional posterior distribution of $\mu$ does not change, however, the conditional posterior distribution of the $\sigma_i^2$ given the $\mu$ can be given as

$$\sigma_i^2|\mu,\mathbf{x}\sim \text{IG}\left(\frac{n_i}{2},\sum_{ij}\frac{(x_{ij}-\mu\ )^2}{2}\right),\ i=1,\cdots,k. \tag{9}$$

**Remark 2**. For the case when $\alpha=1$, prior for $\mu,\sigma_1^2,\sigma_2^2,\ldots,\sigma_k^2$ is $\pi(\mu,\sigma_1^2,\sigma_2^2,\ldots,\sigma_k^2)=\prod_{i=1}^k\sigma_i^{-1}$ which is a right invariant Haar measure [24]. The conditional posterior distribution of $\mu$ remains unchanged, but the conditional posterior distribution of $\sigma_i^2$ given the $\mu$ can be expressed as

$$\sigma_i^2|\mu,\mathbf{x}\sim \text{IG}\left(\frac{n_i-1}{2},\sum_{ij}\frac{(x_{ij}-\mu\ )^2}{2}\right),\ i=1,\cdots,k. \tag{10}$$

# 3. Results

The Gibbs sampling algorithm provides a flexible and widely applicable approach for drawing samples from complex posterior distributions, especially in Bayesian inference where analytical solutions may be intractable. This study used the Gibbs sampling algorithm to approximate the joint distribution from (8) to (9) by iterative sampling from their conditional distributions. Here's a description of the algorithm and its steps:

- **Step 1:** Start with an initial guess $(\mu^{(0)}, \sigma_1^{2(0)}, \cdots, \sigma_k^{2(0)})$. The initial values are chosen as $\mu^{(0)} = \left(\sum_{i=1}^{k} \frac{n_i}{s_i^2}\overline{x}_i\right) / \left(\sum_{i=1}^{k} \frac{n_i}{s_i^2}\right)$ and $\sigma_i^{2(0)}$ as the study sample variance $s_i^2$.

- **Step 2:** Given the $i$th sample $(\mu^{(i)}, \sigma_1^{2(i)}, \cdots, \sigma_k^{2(i)})$, update the $(\mu^{(i+1)})$ from $\pi(\mu|\sigma_1^{2(i)}, \ldots, \sigma_k^{2(i)}, \mathbf{x})$.

- **Step 3:** Generate $\pi(\sigma_1^{2(i+1)}|\mu^{(i)}, \mathbf{x})$.

- **Step 3:** $\vdots$

- **Step 4:** Generate $\pi(\sigma_k^{2(i+1)}|\mu^{(i)}, \mathbf{x})$.

- **Step 5:** Repeat the following steps 2 to 4 until convergence is reached.

## 3.1. Illustrative Examples

We will use two examples to illustrate how the Bayesian method compares to both approximate and exact confidence interval methods. The initial instance originates from Meier's work in 1953 [8], while the second is drawn from the research of Eberhardt et al. in 1989 [9].

### 3.1.1. Example 1

To estimate the percentage of albumin in plasma protein, four separate experiments each employing distinct experimental setups were conducted. The results obtained from these four experiments are given in Table 1 [8].

Table 1: Mean of albumin in plasma protein

| Experiment | $n_i$ | Mean | Variance |
|---|---|---|---|
| A | 12 | 62.30 | 12.99 |
| B | 15 | 60.30 | 7.84 |
| C | 7 | 59.50 | 33.43 |
| D | 16 | 61.50 | 18.51 |

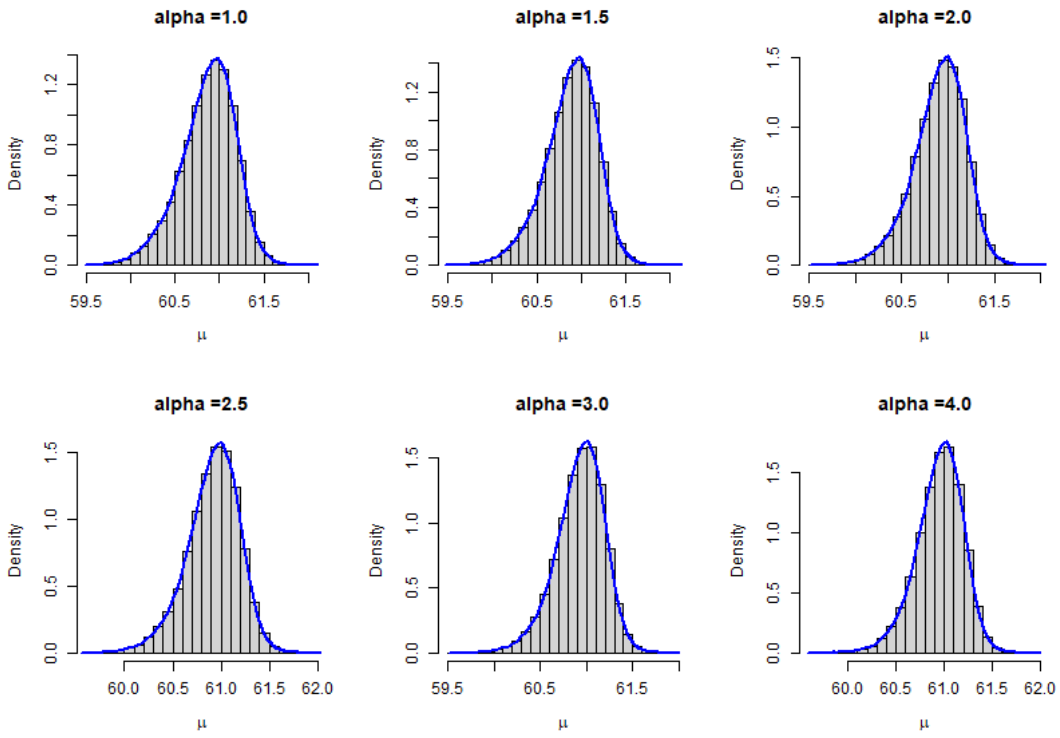


Figure 1: Posterior distribution of the percentage of albumin in plasma protein based on noninformative priors

The mean albumin in plasma protein is estimated and confidence intervals are constructed by four approximations, six exact, and two Bayesian approaches. Table 2 shows the estimated common mean value, with 95% confidence and credible intervals. It should be noted that the Bayesian technique yields a shorter interval than the approximate and exact confidence intervals based on both cases, for this particular problem. Once again, the posterior density credible interval based on prior with $\alpha = 4$ is slightly shorter than its counterpart prior based on other values of $\alpha$. It can be noted that as $\alpha$ increases, the performance based on the Bayesian technique also increases.

Table 2: Interval estimates for $\mu$ of albumin in plasma protein

| **Approximate** | $\mu$ | **95% CI on $\mu$** |
|---|---|---|
| Sinha, $CI_1(\mu)$ | 60.99 | $(59.88; 62.10)$ |
| Meier, $CI_2(\mu)$ | 60.99 | $(59.86; 62.12)$ |
| Large Sample, $CI_3(\mu)$ | 60.99 | $(60.00; 61.98)$ |
| Hartung, $CI_4(\mu)$ | 60.99 | $(59.33; 62.65)$ |
| **Exact** | | **95% CI on $\mu$** |
| Cohen & Sackrowitz, $CI_5(\mu)$ | 60.82 | $(59.14; 62.50)$ |
| Cohen & Sackrowitz, $CI_6(\mu)$ | 60.78 | $(59.20; 62.36)$ |
| Fairweather, $CI_7(\mu)$ | 61.04 | $(59.89; 62.19)$ |
| Jordan & Krishnamoorthy, $CI_8(\mu)$ | 61.00 | $(59.56; 62.44)$ |
| Inverse Normal, $CI_9(\mu)$ | 61.00 | $(59.69; 62.31)$ |
| Fisher, $CI_{10}(\mu)$ | 61.00 | $(59.58; 62.42)$ |
| **Bayesian** | | **95% CrI on $\mu$** |
| $\alpha = 1.0$ | 60.87 | $(60.21; 61.42)$ |
| $\alpha = 1.5$ | 60.89 | $(60.26; 61.42)$ |
| $\alpha = 2.0$ | 60.90 | $(60.32; 61.42)$ |
| $\alpha = 2.5$ | 60.92 | $(60.36; 61.42)$ |
| $\alpha = 3.0$ | 60.93 | $(60.39; 61.41)$ |
| $\alpha = 4.0$ | 60.95 | $(60.45; 61.40)$ |

### 3.1.2. Example 2

In 1989, Eberhardt et al. conducted a study involving four separate experiments focusing on nonfat milk powder and the problem was to draw inference about the mean Selenium in non-fat milk powder by combining the results from four methods [9] (Table 3). The mean Selenium in non-fat milk powder is estimated and confidence intervals

Table 3: Mean Selenium in non-fat milk powder

| Methods | $n_i$ | Mean | Variance |
|---|---|---|---|
| Atomic absorption spectrometry | 8 | 105.00 | 85.71 |
| Neutron activation: | | | |
| 1.) Instrumental | 12 | 109.75 | 20.75 |
| 2.) Radiochemical | 14 | 109.50 | 2.73 |
| Isotope dilution mass spectrometry | 8 | 113.25 | 33.64 |

are constructed by four approximations, six exact, and two Bayesian approaches. Table 4 shows the estimated common mean value, with 95% confidence and credible intervals. It should be noted that the Bayesian technique yields a higher interval than the approximate and exact confidence intervals. The approximation methods by Sinha

(1985) and Meier (1953) yield shorter interval than the Bayesian credible interval. On the other hand, the Inverse Normal exact method also yields a shorter interval than the Bayesian credible interval. Once again, the posterior density credible interval based on prior with $\alpha = 4$ is slightly shorter than its counterpart prior based on other values of $\alpha$.

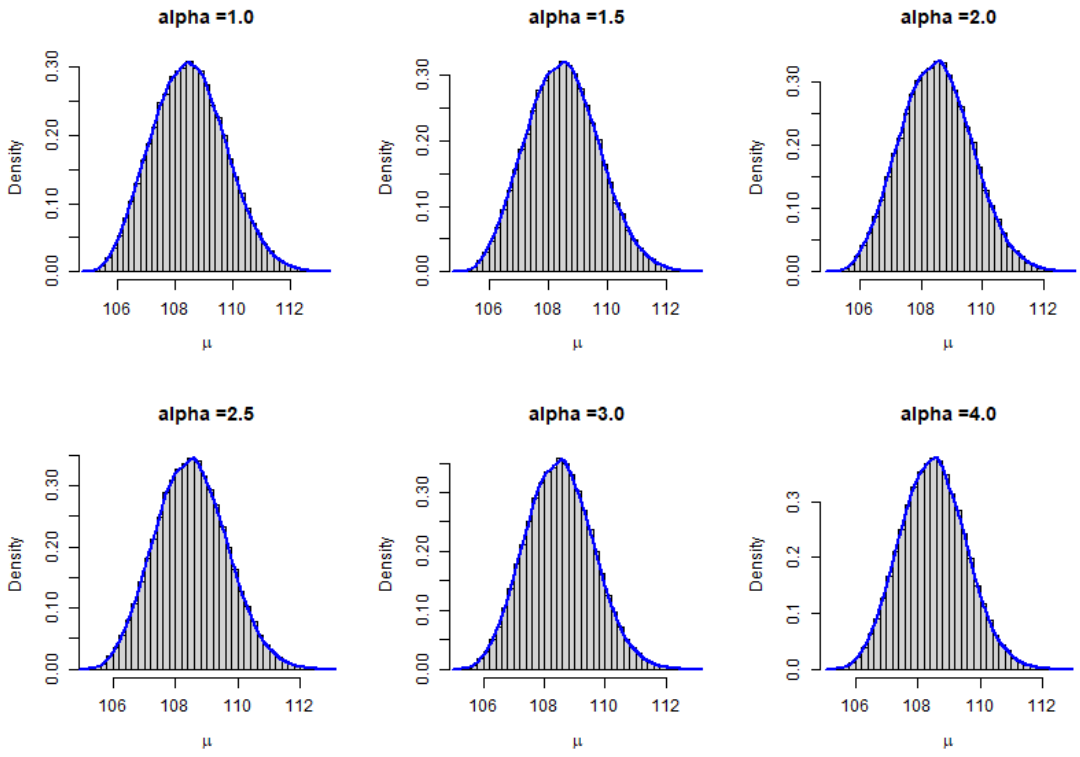


Figure 2: Posterior distribution of the Selenium in non-fat milk powder

Table 4: Interval estimates for $\mu$ of Selenium in non-fat milk powder

| **Approximate** | $\mu$ | **95% CI on** $\mu$ |
|---|---|---|
| Sinha, $CI_1(\mu)$ | 109.60 | $(108.71; 110.49)$ |
| Meier, $CI_2(\mu)$ | 109.60 | $(108.70; 110.50)$ |
| Large Sample, $CI_3(\mu)$ | 109.60 | $(108.80; 110.41)$ |
| Hartung, $CI_4(\mu)$ | 109.60 | $(107.89; 111.31)$ |
| **Exact** | | **95% CI on** $\mu$ |
| Cohen & Sackrowitz, $CI_5(\mu)$ | 109.50 | $(108.12; 110.88)$ |
| Cohen & Sackrowitz, $CI_6(\mu)$ | 109.50 | $(108.23; 110.77)$ |
| Fairweather, $CI_7(\mu)$ | 109.70 | $(108.59; 110.81)$ |
| Jordan & Krishnamoorthy, $CI_8(\mu)$ | 109.60 | $(108.52; 110.68)$ |
| Inverse Normal, $CI_9(\mu)$ | 109.60 | $(108.67; 110.53)$ |
| Fisher, $CI_{10}(\mu)$ | 109.60 | $(108.51; 110.69)$ |
| **Bayesian** | | **95% CrI on** $\mu$ |
| $\alpha = 1.0$ | 108.50 | $(106.10; 110.88)$ |
| $\alpha = 1.5$ | 108.50 | $(106.18; 110.82)$ |
| $\alpha = 2.0$ | 108.50 | $(106.27; 110.75)$ |
| $\alpha = 2.5$ | 108.50 | $(106.30; 110.66)$ |
| $\alpha = 3.0$ | 108.51 | $(106.38; 110.62)$ |
| $\alpha = 4.0$ | 108.51 | $(106.46; 110.48)$ |

## 4. Limitations

While Gibbs sampling is an efficient and widely used method for Bayesian inference, its computational complexity increases significantly with large datasets, which may lead to longer run times or require more advanced computational resources. This limitation could be addressed in future work by exploring alternative sampling methods, such as Hamiltonian Monte Carlo, or by implementing optimization techniques to improve efficiency. Future research could benefit from exploring a wider range of prior distributions, including non-conjugate or more flexible priors. This would allow for a more nuanced modeling of the data and potentially yield improved predictive performance. The model currently assumes normally distributed data, which may not be appropriate for all real-world applications. Future extensions could involve adapting the model to accommodate non-normal populations, such as those with heavy tails or skewness, using robust distributions like t-distributions or skew-normal distributions.

## 5. Conclusion

In conclusion, this study has significantly advanced the field by leveraging Bayesian methodologies to draw inferences concerning the common mean of $k$ different nor-

mal populations. Our approach, which incorporates diverse objective priors, stands out as a notable contribution. Upon comparing with both approximate and exact confidence intervals, it was noted that in one instance, Bayesian methods showcased the effectiveness and reliability of our approach in delivering precise estimates for the common mean. However, this was not replicated in the second scenario. Furthermore, it was discovered that as the parameter $\alpha$ increases, the performance of the Bayesian technique also improves. This work not only expands upon prior research but also underscores the importance of Bayesian techniques in tackling various problems in statistical inference.

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